

# LECTURES ON SUPERSINGULAR K3 SURFACES AND THE CRYSTALLINE TORELLI THEOREM

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**ABSTRACT.** We survey crystalline cohomology, crystals, and formal group laws with an emphasis on geometry. We apply these concepts to K3 surfaces, and especially to supersingular K3 surfaces. In particular, we discuss stratifications of the moduli space of polarized K3 surfaces in positive characteristic, Ogus’ crystalline Torelli theorem for supersingular K3 surfaces, the Tate conjecture, and the unirationality of K3 surfaces.

## INTRODUCTION

In these notes, we cover the following topics

- algebraic deRham cohomology, crystalline cohomology, and  $F$ -crystals,
- characteristic- $p$  aspects of K3 surfaces,
- Ogus’ crystalline Torelli theorem for supersingular K3 surfaces,
- formal group laws, and in particular, the formal Brauer group, and
- unirationality and supersingularity of K3 surfaces.

We assume familiarity with algebraic geometry, say, at the level of the textbooks of Hartshorne [Har77] and Griffiths–Harris [G-H78].

One aim of these notes is to convince the reader that crystals and crystalline cohomology are rather explicit objects, and that they are characteristic- $p$  versions of Hodge structures and deRham cohomology, respectively. Oversimplifying and putting it a little bit sloppily, the *crystalline cohomology* of a smooth and proper variety in characteristic  $p$  is the deRham cohomology of a lift to characteristic zero. (Unfortunately, such lifts may not exist, and even if they do, they may not be unique - it was Grothendieck’s insight that something like crystalline cohomology exists nevertheless and that it is well-defined.) Just as complex deRham cohomology comes with complex conjugation, crystalline cohomology comes with a Frobenius action, and this latter leads to the notion of a *crystal*. Therefore, *period maps* in characteristic  $p$  should take values in moduli spaces of crystals. For example, *Ogus’ crystalline Torelli theorem* states that moduli spaces of certain K3 surfaces,

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namely, *supersingular K3 surfaces*, can be entirely understood via a period map to a moduli space of suitably enriched crystals. Conversely, the classification of crystals arising from K3 surfaces gives rise to a stratification of the moduli space of K3 surfaces in characteristic  $p$ : for example, the *height stratification* in terms of *Newton polygons* arises this way.

A second aim of these notes is to introduce *formal group laws*, and, following Artin and Mazur, to explain how they arise naturally from algebraic varieties. For us, the most important examples will be the *formal Picard group* and the *formal Brauer group* of a smooth and proper variety in characteristic  $p$ . Whereas the former arises as formal completion of the Picard scheme along its origin, the latter does not have such a description, and is something new. Oversimplifying again, we have a sort of crystal associated to a formal group law, namely, its *Cartier–Dieudonné module*. For example, the Cartier–Dieudonné modules of the formal Picard group and the formal Brauer group give rise to subcrystals inside first and second crystalline cohomology of the variety in question. And despite their abstract appearance, these formal group laws do have a geometric interpretation: for example, for supersingular K3 surfaces, the formal Brauer group controls certain very special one-parameter deformations, *moving torsors*, which are characteristic- $p$  versions of twistor space. These deformations are the key to the proof that *supersingular K3 surfaces are unirational*.

And finally, a third aim of these notes is to make some of the more abstract concepts more accessible, which is why we have put an emphasis on computing everything for K3 surfaces. Even for a reader who cares not so much for K3 surfaces, these notes may be interesting, since we show by example, how to perform computations with crystals and formal Brauer groups.

These notes are organized as follows:

- Section 1* We start by discussing deRham cohomology over the complex numbers, and then turn to algebraic deRham cohomology. After a short detour to  $\ell$ -adic cohomology, we introduce the Witt ring  $W$ , and survey crystalline cohomology.
- Section 2* We define K3 surfaces, give examples, and discuss their position within the surface classification. Then, we compute their cohomological invariants, and end by introducing polarized moduli spaces.
- Section 3* Crystalline cohomology takes its values in  $W$ -modules, where  $W$  denotes the Witt ring, and it comes with a Frobenius-action, which leads to the notion of an  $F$ -crystal. After discussing the Dieudonné–Manin classification of  $F$ -crystals up to isogeny, we introduce Hodge and Newton polygons of  $F$ -crystals.
- Section 4* The  $F$ -crystal associated to the second crystalline cohomology group of a K3 surface comes with a quadratic form arising from Poincaré duality, which is captured in the notion of a K3 crystal. After discussing supersingular K3 crystals and the Tate conjecture, we explicitly classify these crystals and construct their moduli space.
- Section 5* Associated to a supersingular K3 surface, we have its supersingular K3 crystal, which gives rise to a period map from the moduli space of supersingular

K3 surfaces to the moduli space of supersingular K3 crystals. Equipping supersingular K3 crystals with ample cones, we obtain a new period map, which is an isomorphism by Ogus' crystalline Torelli theorem.

*Section 6* After defining formal group laws and giving examples, we introduce basic invariants and state the Cartier–Dieudonné classification. Then, we discuss formal group laws arising from algebraic varieties, and relate their Cartier–Dieudonné modules to crystalline cohomology.

*Section 7* We show that a K3 surface is unirational if and only if it is supersingular. The key idea is to give a geometric construction that uses the formal Brauer group.

*Section 8* Finally, we discuss several stratifications of the moduli space of polarized K3 surfaces, which arise from  $F$ -crystals and formal Brauer groups.

For further reading, we refer the interested reader

- to Wedhorn's notes [We08] for more on deRham cohomology and  $F$ -zips, as well as to Illusie's survey [Ill02] for the degeneration of the Frölicher spectral sequence with characteristic- $p$  methods,
- to Chambert-Loir's survey [CL98] for more on crystalline cohomology, and to Katz's article [Ka79] for more on  $F$ -crystals,
- to my own survey [Li13] for more on the classification of surfaces in positive characteristic,
- to Ogus' original articles [Og79] and [Og83] for more on supersingular K3 crystals and the Torelli theorem, and finally,
- to the books of Hazewinkel [Ha78] and Zink [Zi84] for more on formal group laws and their applications.

Of course, this list is by no means complete, but merely represents a small part of the literature for some of the topics touched upon in these notes.

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## 1. CRYSTALLINE COHOMOLOGY

**1.1. Complex Geometry.** Let  $X$  be a smooth and projective variety over the field  $\mathbb{C}$  of complex numbers. In this subsection we briefly recall how the topological, differentiable and holomorphic structure on  $X$  give rise to extra structure on the deRham cohomology groups  $H_{\mathrm{dR}}^*(X/\mathbb{C})$ , and how these structures are related.

First, we consider  $X$  as a differentiable manifold, where differentiable shall always mean differentiable of class  $C^\infty$  without further mentioning it. Let  $\mathcal{A}_X^n$  be the sheaf of  $\mathbb{C}$ -valued differentiable  $n$ -forms with respect to the analytic topology,

and in particular,  $\mathcal{A}_X^0$  is the sheaf of  $\mathbb{C}$ -valued differentiable functions on  $X$ . Then, by the Poincaré lemma, the *differentiable deRham complex*

$$0 \rightarrow \mathcal{A}_X^0 \xrightarrow{d} \mathcal{A}_X^1 \xrightarrow{d} \mathcal{A}_X^2 \xrightarrow{d} \dots$$

is a fine resolution of the constant sheaf  $\underline{\mathbb{C}}$ . Taking global sections, we obtain a complex, whose cohomology is by definition deRham cohomology. And since the differentiable deRham complex is a fine resolution of  $\underline{\mathbb{C}}$ , we obtain a natural isomorphism

$$H^n(X, \underline{\mathbb{C}}) \cong H_{\text{dR}}^n(X/\mathbb{C})$$

for all  $n$ , and refer to [Wa71, Chapter 5] for details and proofs.

Next, we consider  $X$  as an algebraic variety with the Zariski topology. Let  $\Omega_{X/\mathbb{C}}^n$  be the coherent  $\mathcal{O}_X$ -module of Kähler  $n$ -forms, and in particular,  $\Omega_{X/\mathbb{C}}^0$  is the structure sheaf  $\mathcal{O}_X$ . As in the differentiable setting, we obtain a complex of sheaves

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega_{X/\mathbb{C}}^1 \xrightarrow{d} \Omega_{X/\mathbb{C}}^2 \xrightarrow{d} \dots,$$

the *algebraic deRham-complex*. Now, the  $\Omega_{X/\mathbb{C}}^n$  are locally free  $\mathcal{O}_X$ -modules, but not acyclic in general. Therefore, we choose injective resolutions of the  $\Omega_{X/\mathbb{C}}^n$  that are compatible with the differentials of the algebraic deRham complex, and eventually, we obtain a double complex. Then, we take global sections of the injective sheaves of this double complex and take cohomology, which defines *algebraic deRham cohomology*. In particular, algebraic deRham cohomology arises as hypercohomology of the algebraic deRham complex. Since the columns of this double complex compute the coherent cohomology groups  $H^j(X, \Omega_{X/\mathbb{C}}^j)$ , there is a spectral sequence to algebraic deRham cohomology

$$E_1^{i,j} = H^j(X, \Omega_{X/\mathbb{C}}^i) \Rightarrow H_{\text{dR,alg}}^{i+j}(X/\mathbb{C}),$$

the *Frölicher spectral sequence*. We refer to [Ill02] or [We08] for details.

Of course, we can also consider  $X$  as a complex manifold with the analytic topology. Then, we can define the holomorphic deRham complex, which gives rise to a holomorphic Frölicher spectral sequence, see [G-H78, Chapter 3.5] or [Vo02, Chapitre II.8]. By the holomorphic Poincaré lemma, the holomorphic deRham complex is a resolution (although not acyclic) of the constant sheaf  $\underline{\mathbb{C}}$ , which is why the hypercohomology of the holomorphic deRham complex is canonically isomorphic to the differentiable deRham cohomology, see loc. cit. By Serre's GAGA-theorems [Se55], the algebraic and holomorphic  $E_1^{p,q}$ 's are canonically isomorphic, and thus, also the hypercohomologies of the holomorphic and algebraic deRham complex are canonically isomorphic. In particular, the differentiable, holomorphic, and algebraic deRham cohomologies are mutually and canonically isomorphic, and there is only one Frölicher spectral sequence. We will therefore omit the subscript alg from algebraic deRham cohomology in the sequel. Apart from the already given references, we refer the interested reader to [Gr66] for more on this subject.

For complex projective varieties, and even for compact Kähler manifolds, the Frölicher spectral sequence degenerates at  $E_1$  by the Hodge decomposition theorem, see [G-H78, Chapter 0.7] or [Vo02, Théorème 8.28], as well as Exercise 1.1

below. Next, the Frölicher spectral sequence gives rise to a filtration

$$0 = F^{n+1} \subseteq \dots \subseteq F^i \subseteq \dots \subseteq F^0 = H_{\text{dR}}^n(X/\mathbb{C}),$$

the *Hodge filtration*. Since the Frölicher spectral sequence degenerates at  $E_1$ , we obtain canonical isomorphisms

$$F^i/F^{i+1} \cong E_1^{i,n-i} = H^{n-i}(X, \Omega_X^i) \quad \text{for all } i, n \text{ with } 0 \leq i \leq n.$$

Next, we consider  $X$  again only as a differentiable manifold with its analytic topology. Then, sheaf cohomology of the constant sheaf  $\mathbb{Z}$  is isomorphic to singular cohomology

$$H^n(X, \mathbb{Z}) \cong H_{\text{sing}}^n(X, \mathbb{Z}),$$

see, for example, [Wa71, Chapter 5]. In particular, the inclusion  $\mathbb{Z} \subset \mathbb{C}$  gives rise to an isomorphism

$$H_{\text{sing}}^n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H_{\text{dR}}^n(X/\mathbb{C}).$$

and thus, to an integral structure on deRham cohomology. Similarly, the inclusion  $\mathbb{R} \subset \mathbb{C}$  gives rise to a real structure on deRham cohomology. In particular, we obtain a complex conjugation on  $H_{\text{dR}}^n(X/\mathbb{C})$ , which we apply to the Hodge filtration to define a second filtration, the so-called *conjugate filtration*, by setting

$$F_{\text{con}}^i := \overline{F^{n-i}}.$$

It follows from the Hodge decomposition theorem that the Hodge and its conjugate filtration satisfy

$$F^i \cap \overline{F^{n-i}} \cong E_1^{i,n-i} = H^{n-i}(X, \Omega_{X/\mathbb{C}}^i),$$

see [G-H78, Chapter 0.6] or [Vo02, Remarque 8.29]. From this, we deduce

$$H^{n-i}(X, \Omega_{X/\mathbb{C}}^i) = E_1^{i,n-i} \cong \overline{E_1^{n-i,i}} = \overline{H^i(X, \Omega_{X/\mathbb{C}}^{n-i})},$$

for all  $i, n$  with  $0 \leq i \leq n$ . We stress that this isomorphism is induced from complex conjugation, making it a priori not algebraic.

Putting all the results and observations made so far together, we see that each deRham cohomology group  $H_{\text{dR}}^n(X/\mathbb{C})$  is a finite dimensional  $\mathbb{C}$ -vector space together with the following data

- (1) an integral structure,
- (2) a real structure, and in particular, a complex conjugation  $\bar{\phantom{x}}$ , and
- (3) two filtrations  $F^\bullet$  and  $F_{\text{con}}^\bullet$ .

Again, we refer to [Ill02] and [We08] for details and further references.

**1.2. Algebraic deRham cohomology.** Let  $X$  be a smooth and proper variety, but now over a field  $k$  of arbitrary characteristic. The definition of algebraic deRham cohomology and the Frölicher spectral sequence (but not its degeneration at  $E_1$ ) in the previous section is purely algebraic, and thus, we have these cohomology groups and spectral sequences also over  $k$ . Let us now discuss  $E_1$ -degeneration of the Frölicher spectral sequence, as well as extra structures on deRham cohomology in this purely algebraic setting. Although many aspects of this section are discussed

in greater detail in [We08], let us run through the main points needed later on in for the reader's convenience.

**Exercise 1.1.** Let  $X$  be a smooth and proper variety over a field  $k$ . Show that already the existence of the Frölicher spectral sequence implies the inequalities

$$\sum_{i+j=n} h^{i,j}(X) \geq h_{\mathrm{dR}}^n(X) \quad \text{for all } n \geq 1,$$

where  $h^{i,j}(X) = \dim_k H^j(X, \Omega_{X/k}^i)$  and  $h_{\mathrm{dR}}^n(X) = \dim_k H_{\mathrm{dR}}^n(X/k)$ . Moreover, show that equality for all  $n$  is equivalent to the degeneration of the Frölicher spectral sequence at  $E_1$ .

If  $k$  is of characteristic zero, then the Frölicher spectral sequence of  $X$  degenerates at  $E_1$  by the following line of reasoning, which is an instance of the so-called *Lefschetz principle*: namely,  $X$  can be defined over a subfield  $k_0 \subseteq k$  that is finitely generated over  $\mathbb{Q}$ , and then,  $k_0$  can be embedded into  $\mathbb{C}$ . Since cohomology does not change under flat base change, and field extensions are flat, it suffices to prove  $E_1$ -degeneration for  $k = \mathbb{C}$ , where it holds by the results of Section 1.1.

In arbitrary characteristic,  $E_1$ -degeneration of the Frölicher spectral sequence holds, for example, for curves, Abelian varieties, K3 surfaces, and complete intersections, see [We08, (1.5)]. For example, for curves it follows from Theorem 1.2 below, and for K3 surfaces, we will show it in Proposition 2.5.

On the other hand, Mumford [Mu61] gave explicit examples of smooth and projective surfaces  $X$  in positive characteristic  $p$  with non-closed global 1-forms. This means that the exterior derivative  $d : H^0(X, \Omega_{X/k}^1) \rightarrow H^0(X, \Omega_{X/k}^2)$  is non-zero, which gives rise to a non-zero differential in the Frölicher spectral sequence, which implies that it does *not* degenerate at  $E_1$ .

In Section 1.4 below we will recall and construct the ring  $W(k)$  of Witt vectors associated to a perfect field  $k$  of positive characteristic, which is a local, complete, and discrete valuation of ring of characteristic zero with residue field  $k$ , whose unique maximal ideal is generated by  $p$ . In particular, the truncated Witt ring  $W_n(k) := W(k)/(p^n)$  is a flat  $\mathbb{Z}/p^n\mathbb{Z}$ -algebra. For a scheme  $X$  over  $k$ , a *lift* of  $X$  to  $W_n(k)$  is a flat scheme  $\mathcal{X} \rightarrow \mathrm{Spec} W_n(k)$  such that  $\mathcal{X} \times_{\mathrm{Spec} W_n(k)} \mathrm{Spec} k \cong X$ . Such lifts may not exist. However, if they exist, we have the following fundamental result concerning the degeneration of the Frölicher spectral sequence:

**Theorem 1.2** (Deligne–Illusie). *Let  $X$  be a smooth and proper variety over a perfect field  $k$  of characteristic  $p \geq \dim(X)$ , and assume that  $X$  admits a lift to  $W_2(k)$ . Then, the Frölicher spectral sequence of  $X$  degenerates at  $E_1$ .*

This is the main result of [D-I87], and we refer the interested reader to [Ill02] for an expanded version with lots of background information. In fact, this theorem can be used to obtain a purely algebraic proof of the  $E_1$ -degeneration of the Frölicher spectral sequence in characteristic zero [Ill02, Theorem 6.9]. This theorem also shows that the above mentioned examples of Mumford of surfaces with non-closed 1-forms do not lift to  $W_2(k)$ . We refer the interested reader to [Li13, Section 11] for more about liftings, and liftable, as well as non-liftable varieties.

In the previous section, where all varieties were defined over the complex numbers, we applied complex conjugation to the Hodge filtration to obtain a new filtration, the conjugate filtration. Over fields of positive characteristic, there is no complex conjugation. However, since algebraic deRham theory is the hypercohomology of the deRham complex, there exists a second spectral sequence, the *conjugate spectral sequence* (this is only a name in analogy with complex geometry: there is no complex conjugation in positive characteristic)

$$E_2^{i,j} := H^i(X, \mathcal{H}^j(\Omega_{X/k}^\bullet)) \Rightarrow H_{\text{dR}}^{i+j}(X/k),$$

see [G-H78, Chapter 3.5] or [We08, Section 1]. If  $k = \mathbb{C}$ , and when considering  $X$  as a complex manifold and holomorphic differential forms, it follows from the holomorphic Poincaré lemma that the cohomology sheaves  $\mathcal{H}^j(\Omega_{X/\mathbb{C}}^i)$  are zero for all  $j \geq 1$ , and thus, the conjugate spectral sequence induces only a trivial filtration on deRham cohomology.

On the other hand, if  $k$  is of positive characteristic, then the cohomology sheaves  $\mathcal{H}^j(\Omega_{X/k}^i)$  are usually non-trivial for  $q \geq 1$ . More precisely, let  $F : X \rightarrow X'$  be the  $k$ -linear Frobenius morphism. Then, there exists a canonical isomorphism

$$C^{-1} : \Omega_{X'/k}^n \xrightarrow{\cong} \mathcal{H}^n(F_* \Omega_{X/k}^\bullet)$$

for all  $n \geq 0$ , the *Cartier isomorphism*, and we refer to [Ill02, Section 3] or [We08, (1.6)] for details, definitions, and further references. Because of this isomorphism, the conjugate spectral sequence is usually non-trivial in positive characteristic, and gives rise to an interesting second filtration on deRham cohomology, called the *conjugate Hodge filtration*. The data of deRham-cohomology, the Frobenius action, and the two Hodge filtrations is captured in the following structure.

**Definition 1.3.** An  $F$ -zip over a scheme  $S$  of positive characteristic  $p$  is a tuple  $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ , where  $M$  is a locally free  $\mathcal{O}_S$ -module of finite rank,  $C^\bullet$  is a descending filtration on  $M$ ,  $D_\bullet$  is an ascending filtration on  $M$ , and  $\varphi_\bullet$  is a family of  $\mathcal{O}_S$ -linear isomorphisms

$$\varphi_n : (\text{gr}_C^n)^{(p)} \xrightarrow{\cong} \text{gr}_D^n,$$

where  $\text{gr}$  denotes the graded quotient modules, and  $^{(p)}$  denotes Frobenius pullback. The function

$$\begin{aligned} \tau & : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0} \\ n & \mapsto \text{rank}_{\mathcal{O}_S} \text{gr}_C^n \end{aligned}$$

is called the (*filtration*) *type* of the  $F$ -zip.

The category of  $F$ -zips over  $\mathbb{F}_p$ -schemes with only isomorphisms as morphisms forms a smooth Artin-stack  $\mathcal{F}$  over  $\mathbb{F}_p$ , and  $F$ -zips of type  $\tau$  form an open, closed, and quasi-compact substack  $\mathcal{F}^\tau \subseteq \mathcal{F}$ , see [M-W04, Proposition 1.7]. Despite the similarity with period domains for Hodge structures, this moduli space  $\mathcal{F}$  is a rather discrete object, and more of a mod  $p$  reduction of such a period domain. More precisely, if  $k$  is an algebraically closed field of positive characteristic, then the set of  $k$ -rational points of  $\mathcal{F}^\tau$  is finite, and we refer to [M-W04, Theorem 4.4] and [We08, Theorem 3.6] for precise statements.

Thus,  $F$ -zips capture discrete invariants arising from deRham cohomology of smooth and proper varieties in positive characteristic. For example, an  $F$ -zip  $(M, C^\bullet, D_\bullet, \varphi_\bullet)$  is called *ordinary* if the filtrations  $C^\bullet$  and  $D_\bullet$  are in opposition, that is, if the rank  $C^i \cap D_j$  is as small as possible for all  $i, j \in \mathbb{Z}$ . We refer to [M-W04] and [We08] for details, examples, proofs, and further references. Moreover, one can also consider  $F$ -zips with additional structure, such as orthogonal and symplectic forms, see [M-W04, Section 5], as well as [P-W-Z12] for further generalizations.

Finally, we would like to mention that Vasiu independently developed in [Va06], [Va10] a theory similar to the  $F$ -zips of Moonen and Wedhorn [M-W04] within the framework of Shimura  $F$ -crystals modulo  $p$  and truncated Dieudonné modules. We refer the interested reader to [Va06, Sections 3.2.1 and 3.2.9], [Va10], and [X13, Section 2.22] for details, examples, and applications.

**1.3.  $\ell$ -adic cohomology.** The singular cohomology of a smooth and complex projective variety coincides with cohomology in the locally constant sheaf  $\underline{\mathbb{Z}}$  (with respect to the analytic topology). Now, let  $X$  be a smooth variety over a field  $k$  of characteristic  $p \geq 0$ . The Zariski topology on  $X$  is too coarse to give a good cohomology theory for locally constant sheaves. However, for locally constant sheaves of finite Abelian groups, it turns out that étale topology, which is a purely algebraically defined Grothendieck topology that is finer than the Zariski topology, has the desired properties, and we shall equip  $X$  with this topology. It goes without saying that then, being locally constant is then meant with respect to the étale topology. We refer to [Del77] and [Mil80] for precise definitions.

Let  $\ell$  be a prime number, and then, we define  *$\ell$ -adic cohomology*

$$H_{\text{ét}}^n(X, \mathbb{Z}_\ell) := \varprojlim H_{\text{ét}}^n(X, \mathbb{Z}/\ell^m \mathbb{Z}).$$

(When considering locally constant sheaves, étale cohomology works best for finite Abelian groups. So rather than trying to define something like  $H_{\text{ét}}^n(-, \mathbb{Z})$  directly, we take the inverse limit over  $H_{\text{ét}}^n(-, \mathbb{Z}/\ell^m \mathbb{Z})$ , which then results in coefficients in the  $\ell$ -adic numbers  $\mathbb{Z}_\ell$ , see also Section 1.6 below. To deal directly with infinite groups, Bhatt and Scholze [B-S13] introduced the pro-étale topology, but we will not pursue this here.) Next, we define

$$H_{\text{ét}}^n(X, \mathbb{Q}_\ell) := H_{\text{ét}}^n(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

If  $X$  is smooth and proper over an algebraically closed field  $k$ , then this cohomology theory has the following properties:

- (1)  $H_{\text{ét}}^n(X, \mathbb{Q}_\ell)$  is a contravariant functor in  $X$ . The cohomology groups are finite dimensional  $\mathbb{Q}_\ell$ -vector spaces, and zero if  $n < 0$  or  $n > 2 \dim(X)$ .
- (2) There is a cup-product structure

$$\cup_{i,j} : H_{\text{ét}}^i(X, \mathbb{Q}_\ell) \times H_{\text{ét}}^j(X, \mathbb{Q}_\ell) \rightarrow H_{\text{ét}}^{i+j}(X, \mathbb{Q}_\ell).$$

Moreover,  $H_{\text{ét}}^{2 \dim(X)}(X, \mathbb{Q}_\ell)$  is 1-dimensional, and  $\cup_{n, 2 \dim(X)-n}$  induces a perfect pairing, called *Poincaré duality*.

- (3)  $H_{\text{ét}}^n(X, \mathbb{Z}_\ell)$  defines an integral structure on  $H_{\text{ét}}^n(X, \mathbb{Q}_\ell)$ ,



- (4) If  $k = \mathbb{C}$ , one can choose an inclusion  $\mathbb{Q}_\ell \subset \mathbb{C}$  (such inclusions exist using cardinality arguments and the axiom of choice, but they are neither canonical nor compatible with the topologies on these fields), and then, there exist isomorphisms for all  $n$

$$H_{\text{ét}}^n(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong H^n(X, \underline{\mathbb{C}}),$$

where we consider  $X$  as a differentiable manifold on the right hand side and  $\underline{\mathbb{C}}$  is locally constant with respect to the analytic topology. This comparison isomorphism connects  $\ell$ -adic cohomology to de Rham-, singular, and constant sheaf cohomology if  $k = \mathbb{C}$ .

- (5) If  $\text{char}(k) = p > 0$  and  $\ell \neq p$ , then the dimension  $\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^n(X, \mathbb{Q}_\ell)$  is independent of  $\ell$  (see [K-M74]). Thus,

$$b_n(X) := \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^n(X, \mathbb{Q}_\ell)$$

is well-defined for  $\ell \neq p$  and it is called the *n.th Betti number*.

- (6) Finally, there exists a Lefschetz fixed point formula, there are base change formulas, there exist cycle classes in  $H_{\text{ét}}^{2q}(X, \mathbb{Q}_\ell)$  for codimension  $q$  subvarieties,...

We refer to [Har77, Appendix C] for an overview, and to [Mil80] or [Del77] for a thorough treatment. The following example shows that the assumption  $\ell \neq p$  in property (5) above is crucial, and gives a hint of the subtleties involved.

**Example 1.4.** Let  $A$  be a  $g$ -dimensional Abelian variety over an algebraically closed field  $k$  of positive characteristic  $p$ . For a prime  $\ell$ , we define the  *$\ell$ -torsion subgroup scheme*  $A[\ell]$  to be the kernel of multiplication by  $\ell : A \rightarrow A$ . The scheme  $A[\ell]$  is a finite flat group scheme of length  $\ell^{2g}$  over  $k$ , whereas the group of its  $k$ -rational points depends on  $\ell$ :

$$A[\ell](k) \cong \begin{cases} (\mathbb{Z}/\ell\mathbb{Z})^{2g} & \text{if } \ell \neq p, \text{ and} \\ (\mathbb{Z}/p\mathbb{Z})^r & \text{for some } 0 \leq r \leq g \text{ if } \ell = p. \end{cases}$$

This integer  $r$  is called the *p-rank* of  $A$ , and we have

$$\dim_{\mathbb{Q}_\ell} H^1(A, \mathbb{Q}_\ell) = \begin{cases} 2g & \text{if } \ell \neq p, \text{ and} \\ r & \text{if } \ell = p. \end{cases}$$

In particular, the assumption  $\ell \neq p$  in property (5) is crucial, since we have  $r < 2g$  in any case. The group scheme  $A[p]$  is of rank  $p^{2g}$  (although only rank  $p^r$  can be “seen” via  $k$ -rational points), which should be reflected in the “correct”  $p$ -adic cohomology theory. Anticipating crystalline cohomology, which we will introduce in Section 1.5 below, there exists an isomorphism (see [Ill79a, Théorème II.5.2])

$$H_{\text{ét}}^1(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K \cong (H_{\text{cris}}^1(A/W) \otimes_W K)_{[0]} \subset H_{\text{cris}}^1(A/W) \otimes_W K,$$

where the subscript  $[0]$  denotes the slope zero sub- $F$ -isocrystal ( $W$  is the ring of Witt vectors of  $k$ ,  $K$  is its field of fractions, and we refer to Section 3 for details). Since  $H_{\text{cris}}^1(A/W)$  is of rank  $2g$ , it gives the “correct” answer, and even the fact that  $H_{\text{ét}}^1(A, \mathbb{Q}_p)$  is “too small” can be explained using crystalline cohomology. In

fact, we will see in Section 1.6 that there exists no “correct”  $p$ -adic cohomology theory with  $\mathbb{Q}_p$ -coefficients.

**1.4. The ring of Witt vectors.** In the next section, we will introduce crystalline cohomology. Since these cohomology groups are modules over the ring of Witt vectors, let us shortly digress on this ring.

Let  $k$  be a perfect field of positive characteristic  $p$ . For example,  $k$  could be a finite field or algebraically closed. Associated to  $k$ , there exists a ring  $W(k)$ , called the *ring of Witt vectors* (or simply, *Witt ring*) of  $k$ , such that

- (1)  $W(k)$  is a discrete valuation ring of characteristic zero,
- (2) the unique maximal ideal  $\mathfrak{m}$  of  $W(k)$  is generated by  $p$ , and the residue field  $W(k)/\mathfrak{m}$  is isomorphic to  $k$ ,
- (3)  $W(k)$  is complete with respect to the  $\mathfrak{m}$ -adic topology,
- (4) every  $\mathfrak{m}$ -adically complete discrete valuation ring of characteristic zero with residue field  $k$  contains  $W(k)$  as subring,
- (5) the Witt ring  $W(k)$  is functorial in  $k$ .

Note that property (4) shows that  $W(k)$  is unique up to isomorphism. There is also a more intrinsic characterization of the functor  $k \mapsto W(k)$  as left adjoint to reduction modulo  $p$ , but we shall not pursue this here.

Instead, let us quickly run through an explicit construction of  $W(k)$ . We refer to [Se68, Chapitre II.6] and [Ha78, Section 17] for details, proofs, and generalizations. Also, we refer to [C-D13] and [C-D14] for a completely different approach to the ring of Witt vectors. Let  $p$  be a prime. Then, we define the *Witt polynomials* (with respect to  $p$ , which is understood from the context and omitted) to be the following polynomials with coefficients in  $\mathbb{Z}$ :

$$\begin{aligned} W_0(x_0) &:= x_0 \\ W_1(x_0, x_1) &:= x_0^p + px_1 \\ &\dots \\ W_n(x_0, \dots, x_n) &:= \sum_{i=0}^n p^i x_i^{p^{n-i}} = x_0^{p^n} + px_1^{p^{n-1}} + \dots + p^n x_n \end{aligned}$$

Then, one can show that there exist unique polynomials  $S_n$  and  $P_n$  in  $2n + 2$  variables with coefficients in  $\mathbb{Z}$  such that the following holds true:

$$\begin{aligned} W_n(x_0, \dots, x_n) + W_n(y_0, \dots, y_n) &= W_n(S_n(x_0, \dots, x_n, y_0, \dots, y_n)) \\ W_n(x_0, \dots, x_n) \cdot W_n(y_0, \dots, y_n) &= W_n(P_n(x_0, \dots, x_n, y_0, \dots, y_n)) \end{aligned}$$

For an arbitrary ring  $R$ , which is not necessarily of characteristic  $p$ , we define the *truncated Witt ring*  $W_n(R)$  to be the set  $R^n$ , whose ring structure is defined to be

$$\begin{aligned} (x_0, \dots, x_{n-1}) &\oplus (y_0, \dots, y_{n-1}) \\ &:= (S_0(x_0, y_0), \dots, S_{n-1}(x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1})) \\ (x_0, \dots, x_{n-1}) &\odot (y_0, \dots, y_{n-1}) \\ &:= (P_0(x_0, y_0), \dots, P_{n-1}(x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1})) \end{aligned}$$

It turns out that  $W_n(R)$  is indeed a ring with zero  $0 = (0, \dots, 0)$  and unit  $1 = (1, 0, \dots, 0)$ . For example, we have  $S_0(x_0, y_0) = x_0 + y_0$  and  $P_0(x_0, y_0) = x_0 \cdot y_0$ ,

and thus,  $W_1(R)$  is just  $R$  with its usual addition and multiplication. Next, if  $R$  is positive characteristic  $p$ , we define

$$\begin{aligned} V &: (x_0, \dots, x_{n-1}) \mapsto (0, x_0, \dots, x_{n-2}) \\ \sigma &: (x_0, \dots, x_{n-1}) \mapsto (x_0^p, \dots, x_{n-1}^p). \end{aligned}$$

Then,  $V$  is called *Verschiebung* (German for “shift”), and it is an additive map. Next,  $\sigma$  is called *Frobenius*, and it is a ring homomorphism. (In order to avoid a clash of notations when dealing with  $F$ -crystals, see Section 3 below, it is customary to denote the Frobenius on  $W(k)$  by  $\sigma$  rather than  $F$ .) The maps  $V$  and  $\sigma$  are related to multiplication by  $p$  on  $W_n(R)$  by

$$\sigma \circ V = V \circ \sigma = p \cdot \text{id}_{W_n(R)}$$

We have  $W_1(R) = R$  as rings, and for all  $n \geq 2$  the projection  $W_n(R) \rightarrow W_{n-1}(R)$  onto the first  $(n-1)$  components is a surjective ring homomorphism. Then, by definition, the *ring of Witt vectors*  $W(R)$  is the inverse limit

$$W(R) := \varprojlim W_n(R),$$

or, equivalently, the previous construction with respect to the infinite product  $R^{\mathbb{N}}$ .

**Exercise 1.5.** For the finite field  $\mathbb{F}_p$ , show that  $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$  and thus,

$$W(\mathbb{F}_p) \cong \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

is isomorphic to  $\mathbb{Z}_p$ , the ring of  $p$ -adic integers. Show that  $\sigma$  is the identity and  $V$  is multiplication by  $p$  in  $W(\mathbb{F}_p)$ .

**Exercise 1.6.** If  $k$  is a field of positive characteristic, show that  $W(k)$  is a ring of characteristic zero with residue field  $k$ . Set  $\mathfrak{m} := \ker(W(k) \rightarrow k)$  and show that if  $k$  is perfect, then  $\mathfrak{m}$  is generated by  $p$ , that  $W(k)$  is  $\mathfrak{m}$ -adically complete, and that  $W(k)$  is a DVR. On the other hand if  $k$  is not perfect, and if moreover  $k/k^p$  is not a finite field extension, show that  $\mathfrak{m}$  is not finitely generated and that  $W_n(k)$  is not Noetherian for all  $n \geq 2$ .

If  $X$  is a scheme, we can also sheafify the construction of  $W_n(R)$  to obtain sheaves of rings  $W_n\mathcal{O}_X$  and  $W\mathcal{O}_X$ , respectively. The cohomology groups

$$H^i(X, W_n\mathcal{O}_X) \quad \text{and} \quad H^i(X, W\mathcal{O}_X)$$

were introduced and studied by Serre [Se58], they are called *Serre’s Witt vector cohomology* groups, and we will come back to them in Section 6. However, let us already note at this point that the torsion of the  $W(k)$ -module  $H^i(X, W\mathcal{O}_X)$  may *not* be finitely generated (for example, this is the case if  $i = 2$  and  $X$  is a supersingular K3 surface), which is rather unpleasant. Let us finally mention that the  $W_n\mathcal{O}_X$  are just the zeroth step of the *deRham–Witt complex*  $(W_n\Omega_X^j, d)$  introduced by Illusie in [Ill79a], and we refer to [Ill79b] for an overview.

**1.5. Crystalline cohomology.** To a complex projective variety  $X$ , we have its deRham cohomology  $H_{\mathrm{dR}}^*(X/\mathbb{C})$  and showed in Section 1.1 that it comes with extra structure. Now, let  $X$  be a smooth and proper variety over a perfect field  $k$  of positive characteristic  $p$ . Let us shortly summarize what we achieved so far in the algebraic setting:

- (1) In Section 1.2, we associated to  $X$  its algebraic deRham-cohomology, which is a  $k$ -vector space  $H_{\mathrm{dR}}^n(X/k)$  together with a Frobenius action and two filtrations, which is captured in the notion of an  $F$ -zip (Definition 1.3).

On the other hand, there is no integral structure. Another drawback is the following: although there exists a Chern map  $c_1 : \mathrm{Pic}(X) \rightarrow H_{\mathrm{dR}}^2(X/k)$ , we have for every  $\mathcal{L} \in \mathrm{Pic}(X)$

$$c_1(\mathcal{L}^{\otimes p}) = p \cdot c_1(\mathcal{L}) = 0 \quad \text{in} \quad H_{\mathrm{dR}}^2(X/k),$$

giving a zero Chern class even for some very ample line bundles. Also, counting fixed points via Lefschetz fixed point formulas (an important technique when dealing with varieties over finite fields) gives us these numbers as traces in  $k$ , and thus, we obtain the number of fixed points only as a congruence modulo  $p$ .

These observations suggest to look for cohomology theories whose groups are modules over rings of characteristic zero.

- (2) In Section 1.3, we discussed  $\ell$ -adic cohomology  $H_{\mathrm{\acute{e}t}}^n(X, \mathbb{Q}_\ell)$ , which comes with an integral structure from  $H_{\mathrm{\acute{e}t}}^n(X, \mathbb{Z}_\ell)$ , but we have no Hodge filtrations.

Moreover, if  $\ell = p$ , then we have seen in Example 1.4 that  $H_{\mathrm{\acute{e}t}}^n(X, \mathbb{Q}_p)$  does not always give the desired answer. Even worse, Serre showed that there does not exist a “well-behaved” cohomology with  $\mathbb{Q}_p$ -coefficients, and we refer to Section 1.6 for a precise statement.

On the other hand, if  $\ell \neq p$ , then the fields  $k$  and  $\mathbb{Q}_\ell$  usually have little in common, making comparison theorems between deRham- and  $\ell$ -adic cohomology even difficult to conjecture.

It is here, that Witt vectors enter the picture. As we shall now see, crystalline cohomology has all desired features and provides an answer to all problems just raised.

To explain crystalline cohomology, let us assume for a moment that  $X$  is smooth and projective over  $k$  and that there exists a projective *lift* of  $X$  to  $W := W(k)$ , that is, a smooth projective scheme  $\mathcal{X} \rightarrow \mathrm{Spec} W$  such that its special fiber  $\mathcal{X} \times_{\mathrm{Spec} W} \mathrm{Spec} k$  is isomorphic to  $X$ . Then, for each  $n$ , the deRham-cohomology group  $H_{\mathrm{dR}}^n(\mathcal{X}/W)$  is a finitely generated  $W$ -module. It was Grothendieck’s insight [Gr68b] that these cohomology groups are independent of choice of lift  $\mathcal{X}$  of  $X$ . In fact, these cohomology groups can even be defined in case  $X$  does not admit a lift to  $W$ . The construction is quite involved, but we refer to [CL98, Section 1.3] for motivation and to [CL98, Section 2] for a detailed introduction.

More precisely, for every  $m \geq 1$ , we have cohomology groups  $H_{\text{cris}}^*(X/W_m(k))$ , all of which are finitely generated  $W_m(k)$ -modules. For  $m = 1$ , we obtain deRham-cohomology

$$H_{\text{dR}}^n(X/k) \cong H_{\text{cris}}^n(X/W_1(k)) = H_{\text{cris}}^n(X/k) \text{ for all } n \geq 0,$$

and, by definition, the limit

$$H_{\text{cris}}^n(X/W) := H_{\text{cris}}^n(X/W(k)) := \varprojlim H_{\text{cris}}^n(X/W_m(k))$$

is called *crystalline cohomology*. The origin of the name is as follows: although  $X$  may not lift to  $W(k)$ , its cohomology “grows” locally over  $W$ . One can make these growths “rigid”, so to glue and to obtain a well-defined cohomology theory over  $W(k)$ . And thus, growing and being rigid, it is natural to call such an object a “crystal”.

If  $K$  denotes the field of fractions of  $W$ , then it has the following properties:

- (1)  $H_{\text{cris}}^n(X/W)$  is a contravariant functor in  $X$ . These groups are finitely generated  $W$ -modules, and zero if  $n < 0$  or  $n > 2 \dim(X)$ .
- (2) There is a cup-product structure

$$\cup_{i,j} : H_{\text{cris}}^i(X/W)/\text{torsion} \times H_{\text{cris}}^j(X/W)/\text{torsion} \rightarrow H_{\text{cris}}^{i+j}(X/W)/\text{torsion}$$

Moreover,  $H_{\text{cris}}^{2 \dim(X)}(X/W) \cong W$ , and  $\cup_{n, 2 \dim(X)-n}$  induces a perfect pairing, called *Poincaré duality*.

- (3)  $H_{\text{cris}}^n(X/W)$  defines an integral structure on  $H_{\text{cris}}^n(X/W) \otimes_W K$ ,
- (4) If  $\ell$  is a prime different from  $p$ , then (see [K-M74])

$$b_n(X) \stackrel{\text{def}}{=} \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^n(X, \mathbb{Q}_\ell) = \text{rank}_W H_{\text{cris}}^n(X/W),$$

showing that crystalline cohomology computes  $\ell$ -adic Betti numbers.

- (5) If  $X$  lifts to  $W$ , then crystalline cohomology is isomorphic to deRham cohomology of a lift, from which we deduce a universal coefficient formula

$$0 \rightarrow H_{\text{cris}}^n(X/W) \otimes_W k \rightarrow H_{\text{dR}}^n(X/k) \rightarrow \text{Tor}_1^W(H_{\text{cris}}^{n+1}(X/W), k) \rightarrow 0,$$

for all  $n \geq 0$ . This formula also holds true if  $X$  does not lift. In any case, this shows that crystalline cohomology computes deRham cohomology.

- (6) Finally, there exists a Lefschetz fixed point formula, there are base change formulas, there exist cycle classes in  $H_{\text{cris}}^{2q}(X/W)$  for codimension  $q$  sub-varieties,...

By functoriality, the absolute Frobenius morphism  $F : X \rightarrow X$  induces a  $\sigma$ -linear morphism  $\varphi : H_{\text{cris}}^n(X/W) \rightarrow H_{\text{cris}}^n(X/W)$  of  $W$ -modules. Ignoring torsion, this motivates to consider free  $W$ -modules together with injective  $\sigma$ -linear maps, which leads to the notion of an *F-crystal*, to which we come back in Section 3.

We refer the interested reader to [CL98] for a much more detailed introduction to crystalline cohomology, to [Gr68b], [Be74] and [B-O78] for proofs and technical details, as well as to [Ill79a] and [Ill79b] for the connection with the deRham–Witt complex.

**Exercise 1.7.** Let  $X$  be a smooth and proper variety over a perfect field  $k$  of positive characteristic  $p$ , and assume that the Frölicher spectral sequence degenerates at  $E_1$ . Using only the properties of crystalline cohomology mentioned above, show that the following are equivalent

- (1) For all  $n \geq 0$ , the  $W$ -module  $H_{\text{cris}}^n(X/W)$  is torsion-free.
- (2) We have

$$\dim_{\mathbb{Q}_\ell} H_{\text{ét}}^n(X, \mathbb{Q}_\ell) = \dim_k H_{\text{dR}}^n(X/k)$$

for all  $n \geq 0$  and all primes  $\ell \neq p$ .

Thus, the  $p$ -torsion of crystalline cohomology measures the deviation between  $\ell$ -adic Betti numbers to dimensions of deRham-cohomology.

**Examples 1.8.** Let us give a two fundamental examples.

- (1) Let  $A$  be an Abelian variety of dimension  $g$ . Then, all  $H_{\text{cris}}^n(A/W)$  are torsion-free  $W$ -modules. More precisely,  $H_{\text{cris}}^1(A/W)$  is free of rank  $2g$  and for all  $n \geq 2$  there are isomorphisms

$$H_{\text{cris}}^n(A/W) \cong \Lambda^n H_{\text{cris}}^1(A/W).$$

Let us mention the following connection (for those familiar with  $p$ -divisible groups and Dieudonné modules), which we will not need in the sequel: let  $A[p^n]$  be the kernel of multiplication by  $p^n : A \rightarrow A$ , which is a finite flat group scheme of rank  $p^{2gn}$ . By definition, the limit

$$A[p^\infty] := \varprojlim A[p^n]$$

is the  $p$ -divisible group associated to  $A$ . Then, the Dieudonné-module associated to  $A[p^\infty]$  is isomorphic to  $H_{\text{cris}}^1(A/W)$ , compatible with the Frobenius-actions on both sides, see, for example, [Ill79a, Section II.7.1]. We will come back to the Frobenius action on  $H_{\text{cris}}^1(A/W)$  in Section 3.

- (2) For a smooth and proper variety  $X$ , let  $\alpha : X \rightarrow \text{Alb}(X)$  be its Albanese morphism. Then,  $\alpha$  induces an isomorphism

$$H_{\text{cris}}^1(X/W) \cong H_{\text{cris}}^1(\text{Alb}(X)/W).$$

In particular,  $H_{\text{cris}}^1(X/W)$  is always torsion-free. From this, we can compute the crystalline cohomology of curves via their Jacobians. We refer to [Ill79a, Section II.5 and Section II.6] for connections of  $p$ -torsion of  $H_{\text{cris}}^2(X/W)$  with Oda's subspace of  $H_{\text{dR}}^1(X/k)$ , the non-reducedness of the Picard scheme of  $X$ , as well as non-closed 1-forms on  $X$ .

In Section 2, we will compute the crystalline cohomology of a K3 surface.

We already mentioned that Illusie constructed a complex  $(W_m \Omega_{X/k}^j, d)$ , the deRham–Witt complex, and that it coincides with the deRham complex if  $m = 1$ . This complex gives rise to spectral sequences for all  $m \geq 1$

$$E_1^{i,j} := H^j(X, W_m \Omega_{X/k}^i) \Rightarrow H_{\text{cris}}^{i+j}(X/W_m(k)).$$

For  $m = 1$ , this is the Frölicher spectral sequence. In the limit  $m \rightarrow \infty$ , this becomes the *slope spectral sequence* from Hodge–Witt cohomology  $H^j(X, W \Omega_{X/k}^i)$

to crystalline cohomology. Whereas the Frölicher spectral sequence of  $X$  may or may not degenerate at  $E_1$  if  $k$  is of positive characteristic, the slope spectral sequence modulo torsion always degenerates at  $E_1$ . Moreover, the slope spectral sequence (including torsion) degenerates at  $E_1$  if and only if the  $p$ -torsion of all  $H^j(X, W\Omega_{X/k}^i)$  is finitely generated. For  $i = 0$ , this gives a conceptual framework for finite generation of Serre's Witt vector cohomology. We refer to [Ill79a] for details.

Finally, reduction modulo  $p$  gives a map

$$\pi_n : H_{\text{cris}}^n(X/W) \rightarrow H_{\text{dR}}^n(X/k),$$

which, by the universal coefficient formula, is onto if and only if  $H_{\text{cris}}^{n+1}(X/W)$  has no  $p$ -torsion. Thus, if all crystalline cohomology groups are torsion-free  $W$ -modules, then deRham-cohomology is crystalline cohomology modulo  $p$ . Next, by functoriality, the Frobenius of  $X$  induces a  $\sigma$ -linear map  $\varphi : H_{\text{cris}}^*(X/W) \rightarrow H_{\text{cris}}^*(X/W)$ . Under suitable hypotheses on  $X$ , the Frobenius action determines the Hodge- and the conjugate filtration on deRham-cohomology. More precisely, we have the following result of Mazur, and refer to [B-O78, Section 8] details, proofs, and further references.

**Theorem 1.9** (Mazur). *Let  $X$  be a smooth and proper variety over a perfect field  $k$  of positive characteristic  $p$ . Assume that  $H_{\text{cris}}^*(X/W)$  has no  $p$ -torsion, and that the Frölicher spectral sequence of  $X$  degenerates at  $E_1$ . Then,*

$$\begin{array}{llll} \pi_n & \text{maps} & \varphi^{-1}(p^i H_{\text{cris}}^n(X/W)) & \text{onto } F^i \\ \pi_n \circ p^{-i} & \text{maps} & \text{Im}(\varphi) \cap (p^i H_{\text{cris}}^n(X/W)) & \text{onto } F_{\text{con}}^{n-i}, \end{array}$$

where  $F^i$  and  $F_{\text{con}}^{n-i}$  denote the Hodge- and its conjugate filtration on  $H_{\text{dR}}^n(X/k)$ , respectively.

**1.6. Serre's observation.** In this section, we have discussed  $\ell$ -adic and crystalline cohomology, whose groups are  $\mathbb{Q}_\ell$ -vector spaces and  $W(k)$ -modules, respectively. One might wonder, whether crystalline cohomology arises as base change from a cohomology theory, whose groups are  $\mathbb{Z}_p$ -modules, or even, whether all of the above cohomology theories arise from a cohomology theory, whose groups are  $\mathbb{Z}$ -modules or  $\mathbb{Q}$ -vector spaces. Now, cohomology theories that satisfy the “usual” properties discussed in this section are examples of so-called *Weil cohomology theories*, and we refer to [Har77, Appendix C.3] for axioms and discussion.

Serre observed that there exists no Weil cohomology theory in positive characteristic that take values in  $\mathbb{Q}$ -,  $\mathbb{Q}_p$ -, or  $\mathbb{R}$ -vector spaces. In particular, the above question has a negative answer. Here is the sketch of a counter-example: there exist supersingular elliptic curves  $E$  over  $\mathbb{F}_{p^2}$  such that  $\text{End}(E) \otimes \mathbb{Q}$  is a quaternion algebra that is ramified at  $p$  and  $\infty$ . By functoriality, we obtain a non-trivial representation of  $\text{End}(E)$  on  $H^1(E)$ , which, being a Weil cohomology theory, must be 2-dimensional. In particular, we would obtain a non-trivial representation of  $\text{End}(E) \otimes \mathbb{Q}$  in a 2-dimensional  $\mathbb{Q}_p$ - or  $\mathbb{R}$ -vector space, a contradiction. We refer to [Gr68b, p. 315] or [CL98, Section I.1.3] for details.

## 2. K3 SURFACES

**2.1. Definition and examples.** In this section, we turn to K3 surfaces, and will compute the various cohomology groups discussed in the previous section for them. Let us first discuss their position within the classification of surfaces: let  $X$  be a smooth and projective surface over an algebraically closed field  $k$  of characteristic  $p \geq 0$ . Moreover, assume that  $\omega_X$  is numerically trivial, that is,  $\omega_X$  has zero-intersection with every curve on  $X$ . In particular,  $X$  is a minimal surface of Kodaira dimension zero. By the Kodaira–Enriques classification (if  $p = 0$ ) and results of Bombieri and Mumford (if  $p > 0$ ), then  $X$  belongs to one of the following classes:

- (1) Abelian surfaces, that is, Abelian varieties of dimension 2.
- (2) (Quasi-)hyperelliptic surfaces.
- (3) K3 surfaces.
- (4) Enriques surfaces.

We refer to [BHPV, Chapter VI] for the surface classification over  $\mathbb{C}$ , and to [Li13] for an overview in positive characteristic. If  $p \neq 2, 3$ , the only surfaces with  $\omega_X \cong \mathcal{O}_X$  are Abelian surfaces and K3 surfaces. We refer the interested reader to [Li13, Section 7] for some classes of Enriques surfaces in  $p = 2$ , as well as quasi-hyperelliptic surfaces in  $p = 2, 3$  that have trivial canonical sheaves, and to [B-M2] and [B-M3] for a detailed analysis of these surfaces. Here, we are mainly interested in K3 surfaces, and recall the following definition, which holds in any characteristic.

**Definition 2.1.** A *K3 surface* is a smooth and projective surface  $X$  over a field such that

$$\omega_X \cong \mathcal{O}_X \quad \text{and} \quad h^1(X, \mathcal{O}_X) = 0.$$

**Examples 2.2.** Let  $k$  be an algebraically closed field of characteristic  $p \geq 0$ .

- (1) If  $X$  is a smooth quartic surface in  $\mathbb{P}_k^3$ , then  $\omega_X \cong \mathcal{O}_X$  by the adjunction formula, and taking cohomology in the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_k^3}(-4) \rightarrow \mathcal{O}_{\mathbb{P}_k^3} \rightarrow \mathcal{O}_X \rightarrow 0$$

we find  $h^1(\mathcal{O}_X) = 0$ . In particular,  $X$  is a K3 surface.

- (2) Similarly, smooth complete intersections of quadric and cubic hypersurfaces in  $\mathbb{P}_k^4$ , as well as smooth complete intersections of three quadric hypersurfaces in  $\mathbb{P}_k^5$  give examples of K3 surfaces.
- (3) If  $p \neq 2$  and  $A$  is an Abelian surface over  $k$ , then the quotient  $A/\pm \text{id}$  has 16 singularities of type  $A_1$ , and its minimal resolution  $\text{Km}(A)$  of singularities is a K3 surface, the *Kummer surface associated to  $A$* . (We refer the interested reader to [Sh74b] and [K78] to learn what goes wrong if  $p = 2$ , and to [Sch07] how to remedy this.)

We note that these three classes differ in size: the three example classes in (1) and (2) form 19-dimensional families, whereas the Kummer surfaces in (3) form a 3-dimensional family.



**2.2. Cohomological invariants.** Let us now compute the  $\ell$ -adic Betti numbers, the Hodge numbers, and the crystalline cohomology groups of a K3 surface. We will give all details so that the interested reader can see where the characteristic- $p$  proofs are more difficult than the ones in characteristic zero.

**Proposition 2.3.** *The  $\ell$ -adic Betti numbers of a K3 surface are as follows*

$$\begin{array}{c|ccccc} i & 0 & 1 & 2 & 3 & 4 \\ \hline b_i(X) & 1 & 0 & 22 & 0 & 1 \end{array}$$

In particular, we have  $c_2(X) = \sum_{i=0}^4 (-1)^i b_i(X) = 24$ .

PROOF. Since  $X$  is a surface, we have  $b_0 = b_4 = 1$ . By elementary deformation theory of invertible sheaves,  $H^1(\mathcal{O}_X)$  is the Zariski tangent space of  $\text{Pic}_{X/k}^0$  at the origin, see [Se06, Section 3.3], for example. Since  $h^1(\mathcal{O}_X) = 0$  by definition of a K3 surface, it follows that  $\text{Pic}_{X/k}^0$  is trivial. Thus, also the Albanese variety  $\text{Alb}(X)$ , which is the dual of the reduced Picard scheme, is trivial, and we find  $b_1(X) = 2 \dim \text{Alb}(X) = 0$ . By Poincaré duality, we have  $b_1 = b_3 = 0$ . Next, from Noether's formula for surfaces

$$12\chi(\mathcal{O}_X) = c_1(X)^2 + c_2(X),$$

we compute  $c_2(X) = 24$ , which, together with the known Betti numbers, implies  $b_2(X) = 22$ .  $\square$

Next, we recall that the *Hodge diamond* of a smooth projective variety  $Y$  is given by ordering the dimensions  $h^{i,j}(Y) = h^j(Y, \Omega_{Y/k}^i)$  in a rhombus.

**Proposition 2.4.** *The Hodge diamond of a K3 surface is as follows:*

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & h^{1,1} & & h^{0,2} & \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & 20 & & 1 & \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

PROOF. We have  $h^{0,0} = h^{2,2} = 1$  since  $X$  is a surface, and  $h^{0,1} = 0$  by the definition of a K3 surface. Next, Serre duality gives  $h^{0,1} = h^{2,1}$  and  $h^{1,0} = h^{1,2}$ . If  $k = \mathbb{C}$ , then complex conjugation induces the Hodge symmetry  $h^{1,0} = h^{0,1}$ . However, in positive characteristic, this Hodge symmetry may fail in general (see [Se58] and [Li08] for examples), and thus, we have to compute  $h^{1,0}(X)$  another way: using the isomorphism  $\mathcal{E}^\vee \cong \mathcal{E} \otimes \det(\mathcal{E})$ , which holds for locally free sheaves of rank 2 (see [Har77, Exercise II.5.16], for example), we find  $T_X \cong \Omega_{X/k}^1$  for a K3 surface, and thus

$$H^{1,0}(X) \stackrel{\text{def}}{=} H^0(\Omega_{X/k}^1) \cong H^0(T_X).$$

Now, by a theorem of Rudakov and Shafarevich [R-S76], a K3 surface has no non-zero global vector fields, and thus, these cohomology groups are zero. Finally, we

use the Grothendieck–Hirzebruch–Riemann–Roch theorem to compute

$$\begin{aligned} & \chi(\Omega_{X/k}^1) \\ &= \text{rank}(\Omega_{X/k}^1) \cdot \chi(\mathcal{O}_X) + \frac{1}{2} \left( c_1(\Omega_{X/k}^1) \cdot (c_1(\Omega_{X/k}^1) - K_X) \right) - c_2(\Omega_{X/k}^1) \\ &= 4 + 0 - 24 = -20, \end{aligned}$$

which implies  $h^1(\Omega_{X/k}^1) = 20$ .  $\square$

As a consequence of this proposition, together with the Rudakov–Shafarevich theorem on non-existence of global vector fields on K3 surfaces, we obtain

**Proposition 2.5.** *For a K3 surface  $X$ , the Frölicher spectral sequence*

$$E_1^{i,j} = H^j(X, \Omega_{X/k}^i) \Rightarrow H_{\text{dR}}^{i+j}(X/k)$$

*degenerates at  $E_1$ . Moreover,  $H_{\text{cris}}^n(X/W)$  is a free  $W$ -module of rank  $b_n(X)$  for all  $n \geq 0$ .*

**PROOF.** By Proposition 2.4 and using the isomorphism  $\Omega_X^1 \cong T_X$  (seen in the proof of Proposition 2.4), we have  $H^2(T_X) \cong H^2(\Omega_X^1) = H^{1,2} = 0$ . Since  $H^2(T_X) = 0$ , deformations of  $X$  are unobstructed, and thus,  $X$  lifts to  $W_2(k)$  (see also the discussion in Section 2.3 and Theorem 2.7 below), and thus, degeneracy of the Frölicher spectral sequence at  $E_1$  follows from Theorem 1.2. From Proposition 2.4 and Exercise 1.1 we compute the dimensions of the deRham cohomology groups, which then turn out to be the same as the  $\ell$ -adic Betti numbers given in Proposition 2.3. Thus, by Exercise 1.7, the crystalline cohomology groups are free  $W$ -modules of the stated rank.  $\square$

**Remark 2.6.** For a smooth and proper variety  $X$  over a perfect field  $k$ , the slope spectral sequence from Hodge–Witt to crystalline cohomology degenerates at  $E_1$  if and only if all the  $W$ -modules  $H^j(X, W\Omega_{X/k}^i)$  are finitely generated [Ill79a, Théorème II.3.7]. For a K3 surface, this is the case if and only if it is not supersingular – we refer to Section 4 for definition of supersingularity and to [Ill79a, Section II.7.2] for details.

**2.3. Deformation theory.** Infinitesimal and formal deformations of a smooth and proper variety  $X$  over a field  $k$  can be controlled by a tangent–obstruction theory arising from the  $k$ -vector spaces  $H^i(T_X)$ ,  $i = 0, 1, 2$ , see [Se06, Chapter 2] or [F-G05, Chapter 6] for a reader-friendly introduction.

Let us recall the most convenient case: if  $H^2(T_X) = 0$ , then every infinitesimal deformation of order  $n$  can be extended to one of order  $n + 1$ , and then, the set of all such extensions is an affine space under  $H^1(T_X)$ . In particular, this applies to lifting problems: if  $k$  is perfect of positive characteristic, then the Witt ring  $W(k)$  is a limit of rings  $W_n(k)$ , see Section 1.4. Since the kernel of  $W_{n+1}(k) \rightarrow W_n(k)$  is the ideal generated by  $p^{n-1}$  and  $(p^{n-1})^2 = 0$ , it is a small extension, and thus, a smooth and proper variety  $X$  over a perfect field  $k$  of positive characteristic with  $H^2(T_X) = 0$  admits a formal lift to  $W(k)$ , and we refer to [Ill05, Chapter 8.5] and [Li13, Section 11.2] for details and further references. Since this most convenient case applies to K3 surfaces, we have the following result.

**Theorem 2.7.** *Let  $X$  be a K3 surface over a perfect field  $k$  of positive characteristic. Then, the formal deformation space  $\mathrm{Def}(X)$  of  $X$  is smooth of relative dimension 20 over  $W(k)$ , that is,*

$$\mathrm{Def}(X) \cong \mathrm{Spf} W(k)[[t_1, \dots, t_{20}]].$$

*In particular,  $X$  formally lifts over  $W(k)$ .*

PROOF. By Proposition 2.4 and using the isomorphism  $\Omega_X^1 \cong T_X$  (seen in the proof of Proposition 2.4), we find

$$h^0(T_X) = h^2(T_X) = 0 \quad \text{and} \quad h^1(T_X) = 20,$$

from which all assertions follow from standard results of deformation theory, see [Se06, Chapter 2] or [F-G05, Chapter 6], for example.  $\square$

If  $X$  is a K3 surface over a perfect field  $k$  of positive characteristic, then the previous theorem implies that there exists a compatible system  $\{\mathcal{X}_n \rightarrow \mathrm{Spec} W_n(k)\}_n$  of algebraic schemes  $\mathcal{X}_n$ , each flat over  $W_n(k)$ , and with special fiber  $\mathcal{X}_1 = X$ . Now, the limit of this system is a formal scheme [Har77, Section II.9], and it is not clear whether it is algebraizable, that is, we do not know, whether this limit arises as completion of a scheme over  $W(k)$  along its special fiber (in fact, it is not true in general, see Section 2.4 below).

By Grothendieck's existence theorem (see [Ill05, Theorem 8.4.10], for example), algebraization of formal schemes holds, for example, if one is able to equip  $\mathcal{X}_n$  with a compatible system  $\mathcal{L}_n$  of invertible sheaves on  $\mathcal{X}_n$  such that  $\mathcal{L}_1$  is ample on  $\mathcal{X}_1 = X$ . This poses the question whether a given formal deformation can be equipped with such a compatible system of invertible sheaves. The obstruction to deforming an invertible sheaf to a small extension lies in  $H^2(\mathcal{O}_X)$ , which is 1-dimensional for a K3 surface. We thus expect that this should impose one non-trivial equation to  $\mathrm{Def}(X)$ , which is true and made precise by the following results of Deligne, [Del81a, Proposition 1.5] and [Del81a, Théorème 1.6].

**Theorem 2.8** (Deligne). *Let  $X$  be a K3 surface over a perfect field  $k$  of positive characteristic, and let  $\mathcal{L}$  be a non-trivial invertible sheaf on  $X$ . Then, the space  $\mathrm{Def}(X, \mathcal{L})$  of formal deformations of the pair  $(X, \mathcal{L})$  is a formal Cartier divisor inside  $\mathrm{Def}(X)$ , that is,*

$$\mathrm{Def}(X, \mathcal{L}) \subset \mathrm{Def}(X),$$

*is a formal subscheme defined by one equation. Moreover,  $\mathrm{Def}(X, \mathcal{L})$  is flat and of relative dimension 19 over  $W(k)$ .*

Unfortunately, it is not clear whether  $\mathrm{Def}(X, \mathcal{L})$  is smooth over  $W(k)$ , and we refer to [Og79, §2] for an analysis of its singularities. In particular, if we pick an ample invertible sheaf  $\mathcal{L}$  on  $X$  in order to construct a formal lift of the pair  $(X, \mathcal{L})$  to  $W(k)$  (in order to apply Grothendieck's existence theorem), then it could happen that  $\mathrm{Def}(X, \mathcal{L})$  is flat, but not smooth over  $W(k)$ . Thus, a priori, we only have an algebraic lift of  $X$  to some finite extension ring  $R \supseteq W(k)$ . However, thanks to a refinement of Ogus [Og79, Corollary 2.3] of Deligne's result, we have

**Theorem 2.9** (Deligne, Ogus). *Let  $X$  be a K3 surface over an algebraically closed field of odd characteristic. Then, there exists a projective lift of  $X$  to  $W(k)$ .*

PROOF. By [Og79, Corollary 2.3], any nonsuperspecial K3 surface can be lifted projectively to  $W(k)$ , and we refer to [Og79, Example 1.10] for the notion of superspecial K3 surfaces. Since the Tate-conjecture holds for K3 surfaces in odd characteristic (see Theorem 4.6 below), the only nonsuperspecial K3 surface is the supersingular K3 surface with Artin invariant  $\sigma_0 = 1$ , see [Og79, Remark 2.4]. However, this latter surface is the Kummer surface associated to the self-product of a supersingular elliptic curve by [Og79, Corollary 7.14] and can be lifted “by hand” projectively to  $W(k)$ .  $\square$

**2.4. Moduli spaces.** By Theorem 2.7, the formal deformation space  $\text{Def}(X)$  of a K3 surface  $X$  is formally smooth and 20-dimensional over  $W(k)$ . However, it is not clear (and in fact, not true) whether all formal deformations are algebraizable. By a theorem of Zariski and Goodman (see [Ba01, Theorem 1.28], for example), a smooth and proper algebraic surface is automatically projective, which applies in particular to K3 surfaces. Thus, picking an ample invertible sheaf  $\mathcal{L}$  on an algebraic K3 surface  $X$ , there is a formal Cartier divisor  $\text{Def}(X, \mathcal{L}) \subset \text{Def}(X)$  by Theorem 2.8, along which  $\mathcal{L}$  extends. Since formal and polarized deformations are algebraizable by Grothendieck’s existence theorem (see Section 2.3), we can algebraize the 19-dimensional formal family over  $\text{Def}(X, \mathcal{L})$ . Using Artin’s approximation theorems, this latter family can be descended to a polarized family of K3 surfaces that is 19-dimensional and of finite type over  $W(k)$ , and one may think of it as an étale neighborhood of  $(X, \mathcal{L})$  inside a moduli space of suitably polarized K3 surfaces. In fact, this can be made precise to give a rigorous algebraic construction for moduli spaces of polarized K3 surfaces, and we refer to [Ar74b, Section 5] and [Ri06] for technical details. These moduli spaces of polarized K3 surfaces are 19-dimensional, whereas the unpolarized formal deformation space is 20-dimensional.

Now, before proceeding, let us shortly leave the algebraic world: over the complex numbers, there exists a 20-dimensional analytic moduli space for compact Kähler surfaces that are of type K3, and most of which are not algebraic. Moreover, this moduli space is smooth, but not Hausdorff. Inside it, the set of algebraic K3 surfaces is a countable union of analytic divisors. In fact, these divisors correspond to moduli spaces of algebraic and polarized K3 surfaces. We refer to [BHPV, Chapter VIII] for details and further references. We mention this to convince the reader that it is not possible to obtain a 20-dimensional moduli space of algebraic K3 surfaces, even over the complex numbers.

Therefore, when considering moduli of algebraic K3 surfaces, one usually looks at moduli spaces of (primitively) polarized surfaces. Here, an invertible sheaf  $\mathcal{L}$  on a variety  $X$  is called *primitive* if it is not of the form  $\mathcal{M}^{\otimes k}$  for some  $k \geq 2$ . Then,

for a ring  $R$ , we consider the functor  $\mathcal{M}_{2d,R}^\circ$

$$\begin{aligned} \left( \begin{array}{c} \text{schemes} \\ \text{over } R \end{array} \right) & \rightarrow (\text{groupoids}) \\ S & \mapsto \left\{ \begin{array}{l} \text{flat morphisms of algebraic spaces } (\mathcal{X}, \mathcal{L}) \rightarrow S, \\ \text{all of whose geometric fibers are K3 surfaces, and} \\ \text{such that } \mathcal{L} \text{ restricts to a primitive polarization of} \\ \text{degree (=self-intersection) } 2d \text{ on each fiber} \end{array} \right\} \end{aligned}$$

In view of the above discussion it should not be surprising that this functor is representable by a separated Deligne–Mumford stack [Ri06, Theorem 4.3.3].

Combining [Ri06, Proposition 4.3.11], [Ma12, Section 5], and [MP13, Corollary 4.16], we even have the following results on the global geometry of these moduli spaces. Before stating them, let us mention that (2) and (3) are neither obvious nor straight forward, and partly rest on the Kuga–Satake construction, which allows to obtain results about moduli spaces of K3 surfaces from the corresponding results for moduli spaces of Abelian varieties.

**Theorem 2.10** (Madapusi-Pera, Maulik, Rizov). *The Deligne–Mumford stack*

- (1)  $\mathcal{M}_{2d, \mathbb{Z}[\frac{1}{2d}]}^\circ$  is smooth over  $\mathbb{Z}[\frac{1}{2d}]$ .
- (2)  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  is quasi-projective over  $\mathbb{F}_p$  if  $p \geq 5$  and  $p \nmid d$ .
- (3)  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  is geometrically irreducible over  $\mathbb{F}_p$  if  $p \geq 3$  and  $p^2 \nmid d$ .

In Section 8 below, we will introduce and discuss a stratification of  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$ , which only exists in positive characteristic. It would be interesting to understand the geometry and the singularities (if there are any) of  $\mathcal{M}_{2d, \mathbb{F}_p}$  if  $p$  divides  $2d$ .

### 3. F-CRYSTALS

In Section 1.5, we introduced crystalline cohomology, and we computed it for K3 surfaces in Section 2.2. Just as Hodge structures abstractly capture the linear algebra data coming from deRham-cohomology of a smooth complex projective variety (see the end of Section 1.1), *F-crystals* capture the semi-linear data coming from crystalline cohomology. In this section, after introducing *F-crystals*, we associate to them two polygons: the *Hodge polygon* and the *Newton polygon*. Under the assumptions of Theorem 1.9 (which hold, for example, for K3 surfaces), the *F-crystal* associated to  $H_{\text{cris}}^*(X/W)$  not only computes deRham-cohomology, but also the Hodge and its conjugate filtration. On the other hand, by a result of Dieudonné and Manin, we can classify *F-crystals* up to *isogeny* in terms of *slopes*, which gives rise to a second polygon, the *Newton polygon*, which always lies on or above the Hodge polygon. The deviation between Hodge and Newton polygon gives rise to new discrete invariants of varieties in positive characteristic that have no analog in characteristic zero. In Section 8, we will use these discrete invariants to stratify the moduli space of K3 surfaces.

**3.1. F-crystals.** In Section 1.4, we introduced and discussed the Witt ring  $W := W(k)$  for a perfect field  $k$  of positive characteristic  $p$ . We denote by  $K$  its field of fractions. We also recall that the Frobenius morphism  $x \mapsto x^p$  of  $k$  induces a ring homomorphism  $\sigma : W \rightarrow W$  by functoriality, and that there exists an additive map  $V : W \rightarrow W$  such that  $p = \sigma \circ V = V \circ \sigma$ . In particular,  $\sigma$  is injective.

**Definition 3.1.** An *F-crystal*  $(M, \varphi_M)$  over  $k$  is a free  $W$ -module  $M$  of finite rank together with an injective and  $\sigma$ -linear map  $\varphi_M : M \rightarrow M$ , that is,  $\varphi_M$  is additive, injective, and satisfies

$$\varphi_M(r \cdot m) = \sigma(r) \cdot \varphi_M(m) \quad \text{for all } r \in W, m \in M.$$

An *F-isocrystal*  $(V, \varphi_V)$  is a finite dimensional  $K$ -vector space  $V$  together with an injective and  $\sigma$ -linear map  $\varphi_V : V \rightarrow V$ .

A *morphism*  $u : (M, \varphi_M) \rightarrow (N, \varphi_N)$  of *F-crystals* (resp., *F-isocrystals*) is a  $W$ -linear (resp.,  $K$ -linear) map  $M \rightarrow N$  such that  $\varphi_N \circ u = u \circ \varphi_M$ . An *isogeny* of *F-crystals* is a morphism  $u : (M, \varphi_M) \rightarrow (N, \varphi_N)$  of *F-crystals*, such that the induced map  $M \otimes_W K \rightarrow N \otimes_W K$  is an isomorphism of *F-isocrystals*.

Let us give two examples of *F-crystals*, one arising from geometry (and being the prototype of such an object), the other one purely algebraic (and being crucial for the isogeny classification later on).

**Example 3.2.** Let  $X$  be a smooth and proper variety over  $k$ . Then, for every  $n \geq 0$ ,

$$H^n := H_{\text{cris}}^n(X/W)/\text{torsion}$$

is a free  $W$ -module of finite rank. By functoriality, the absolute Frobenius morphism  $F : X \rightarrow X$  induces a  $\sigma$ -linear map  $\varphi : H^n \rightarrow H^n$ . Next, Poincaré duality induces a perfect pairing

$$\langle -, - \rangle : H^n \times H^{2 \dim(X) - n} \rightarrow H^{2 \dim(X)}(X/W) \cong W,$$

which satisfies the following compatibility with Frobenius

$$\langle \varphi(x), \varphi(y) \rangle = p^{\dim(X)} \cdot \sigma \langle x, y \rangle.$$

Since  $\sigma$  is injective on  $W$ , it follows that also  $\varphi : H^n \rightarrow H^n$  is injective, and thus,  $(H^n, \varphi)$  is an *F-crystal*.

**Example 3.3.** Let  $W_\sigma \langle T \rangle$  be the non-commutative polynomial ring in the variable  $T$  over  $W$  subject to the relations

$$T \cdot r = \sigma(r) \cdot T \quad \text{for all } r \in W.$$

Let  $\alpha = r/s \in \mathbb{Q}_{\geq 0}$ , where  $r, s$  are non-negative and coprime integers. Then,

$$M_\alpha := W_\sigma \langle T \rangle / (T^s - p^r)$$

together with  $\varphi : m \mapsto T \cdot m$  defines an *F-crystal*  $(M_\alpha, \varphi)$  of rank  $s$ . The rational number  $\alpha$  is called the *slope* of  $(M_\alpha, \varphi)$ .

The importance of the previous example comes from the following result, which classifies *F-crystals* over algebraically closed fields up to isogeny:

**Theorem 3.4** (Dieudonné–Manin). *Let  $k$  be an algebraically closed field of positive characteristic. Then, the category of  $F$ -crystals over  $k$  up to isogeny is semi-simple and the simple objects are the  $(M_\alpha, \varphi)$ ,  $\alpha \in \mathbb{Q}_{\geq 0}$  from Example 3.3.*

We note in passing that not every  $F$ -isocrystal is of the form  $M \otimes_{W(k)} K$  for some  $F$ -crystal  $(M, \varphi)$ . Those  $F$ -isocrystals that do are called *effective*.

**Definition 3.5.** Let  $(M, \varphi)$  be an  $F$ -crystal over an algebraically closed field  $k$  of positive characteristic, and let

$$(M, \varphi) \sim \bigoplus_{\alpha \in \mathbb{Q}_{\geq 0}} M_\alpha^{n_\alpha}$$

be its decomposition up to isogeny according to Theorem 3.4. Then, the elements in the set

$$\{\alpha \in \mathbb{Q}_{\geq 0} \mid n_\alpha \neq 0\}$$

are called the *slopes* of  $(M, \varphi)$ . For every slope  $\alpha$  of  $(M, \varphi)$ , the integer

$$\lambda_\alpha := n_\alpha \cdot \text{rank}_W M_\alpha$$

is called the *multiplicity* of the slope  $\alpha$ . In case  $(M, \varphi)$  is an  $F$ -crystal over a perfect field  $k$ , we define its slopes and multiplicities to be the ones of  $(M, \varphi) \otimes_{W(k)} W(\bar{k})$ , where  $\bar{k}$  is an algebraic closure of  $k$ .

**3.2. Newton and Hodge polygons.** Let  $(M, \varphi)$  be an  $F$ -crystal. We order its slopes in ascending order

$$0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_t$$

and denote by  $\lambda_1, \dots, \lambda_t$  the respective multiplicities.

Then, the *Newton polygon* of  $(M, \varphi)$  is defined to be the graph of the piecewise linear function  $\text{Nwt}_M$  from the interval  $[0, \text{rank } M] \subset \mathbb{R}$  to  $\mathbb{R}$ , such that  $\text{Nwt}_M(0) = 0$  and whose graph has the following slopes

$$\begin{array}{ll} \text{slope} & \alpha_1 \quad \text{if } 0 \leq t < \lambda_1, \\ \text{slope} & \alpha_2 \quad \text{if } \lambda_1 \leq t < \lambda_1 + \lambda_2, \\ & \dots \end{array}$$

Since we ordered the slopes in ascending order, this polygon is convex. Next, it follows easily from the definitions that the vertices of this polygon have integral coordinates. Clearly, since the Newton polygon is built from slopes, it only depends on the isogeny class of  $(M, \varphi)$ . Conversely, we can read off all slopes and multiplicities from the Newton polygon, and thus, the Newton polygon actually determines the  $F$ -crystal up to isogeny.

Next, we define the *Hodge polygon* of  $(M, \varphi)$ , whose definition is motivated by Theorem 1.9, but see also Theorem 3.8 below. Since  $\varphi$  is injective,  $M/\varphi(M)$  is an Artinian  $W$ -module, and thus, there exist non-negative integers  $h_i$  and an isomorphism

$$M/\varphi(M) \cong \bigoplus_{i \geq 1} (W/p^i W)^{h_i}.$$

Moreover, we define

$$h_0 := \text{rank } M - \sum_{i \geq 1} h_i.$$

Then, the Hodge polygon of  $(M, \varphi)$  is defined to be the graph of the piecewise linear function  $\text{Hdg}_M$  from the interval  $[0, \text{rank } M] \subset \mathbb{R}$  to  $\mathbb{R}$ , such that  $\text{Hdg}_M(0) = 0$  and whose graph has the following slopes

$$\begin{array}{ll} \text{slope} & 0 \quad \text{if } 0 \leq t < h_0, \\ \text{slope} & 1 \quad \text{if } h_0 \leq t < h_0 + h_1, \\ & \dots \end{array}$$

As above with the Newton polygon, the Hodge polygon is convex and its vertices have integral coordinates.

**Example 3.6.** Let  $M_\alpha$  with  $\alpha = r/s \in \mathbb{Q}_{\geq 0}$  be the  $F$ -crystal from Example 3.3. Then,

$$M_\alpha / \varphi(M_\alpha) \cong W/p^r W,$$

we obtain  $h_0 = s - 1$ ,  $h_r = 1$ , and  $h_i = 0$  for  $i \neq 0, r$ .

Now, we assume that  $0 < \alpha < 1$ , that is,  $0 < r < s$ , and we define a  $W$ -module  $N_\alpha$  together with an embedding into  $M_\alpha \otimes_W K$  as follows:

$$\begin{array}{ccc} N_\alpha := W[T, U] / (TU - p, T^{s-r} - U^r) & \rightarrow & M_\alpha \otimes_W K \\ T & \mapsto & T \\ U & \mapsto & pT^{-1} = p^{1-r}T^{s-1}. \end{array}$$

Then,  $N_\alpha$  inherits the structure of an  $F$ -crystal from  $M_\alpha \otimes_W K$ , and it is isogenous to  $M_\alpha$ . Since

$$N_\alpha / \varphi(N_\alpha) \cong (W/pW)^r,$$

we obtain  $h_0 = s - r$ ,  $h_1 = r$ , and  $h_i = 0$  for  $i \neq 0, 1$ . In particular, we find the following Hodge polygons (solid lines) and Newton polygons (dotted lines):



This example shows that the Hodge polygon, unlike the Newton polygon, is in general *not* an isogeny invariant of the  $F$ -crystal  $(M, \varphi)$ . However, the isogeny class of an  $F$ -crystal, that is, its Newton polygon, puts restrictions on the possible Hodge polygons:

**Proposition 3.7.** *Let  $(M, \varphi)$  be an  $F$ -crystal. Then, its Newton-polygon lies on or above its Hodge polygon, and both have the same startpoint and endpoint:*

$$\begin{array}{ll} \text{Nwt}_M(t) & \geq \text{Hdg}_M(t) \quad \text{for all } t \in [0, \text{rank } M], \text{ and} \\ \text{Nwt}_M(t) & = \text{Hdg}_M(t) \quad \text{for } t_0 = 0 \text{ and } t = \text{rank } M. \end{array}$$



**3.3. F-crystals arising from geometry.** Now, we link the Hodge polygon of an  $F$ -crystal that arises from crystalline cohomology of a variety to the Hodge numbers of that variety, which justifies the terminology: let  $X$  be a smooth and proper variety over  $k$  and fix an integer  $n \geq 0$ . Then, we consider the Hodge numbers

$$\tilde{h}_i := h^{i, n-i} = \dim_k H^{n-i}(X, \Omega_{X/k}^i) \quad \text{for all } 0 \leq i \leq n$$

and, as before with the Hodge polygon, we construct from these integers a piecewise linear function

$$\widetilde{\text{Hdg}}_X^n : [0, \tilde{h}_{\text{dR}}^n] \rightarrow \mathbb{R}, \quad \text{where} \quad \tilde{h}_{\text{dR}}^n = \sum_{i=0}^n \tilde{h}_i,$$

and whose associated convex polygon is called the *geometric Hodge polygon*.

The following deep and important result shows that under extra hypotheses the  $F$ -crystal associated to the crystalline cohomology of a smooth and proper variety detects its Hodge numbers. In fact, part of this result just rephrases Theorem 1.9 in terms of Hodge polygons.

**Theorem 3.8** (Mazur, Nygaard, Ogus). *Let  $X$  be a smooth and proper variety over a perfect field  $k$  of positive characteristic. Fix an integer  $n \geq 0$  and let*

$$H^n := (H_{\text{cris}}^n(X/W)/\text{torsion}, \varphi)$$

*be the associated  $F$ -crystal. Then*

- (1) *For all  $t \in [0, \text{rank } H^n]$ , we have*

$$\text{Nwt}_{H^n}(t) \geq \widetilde{\text{Hdg}}_X^n(t).$$

- (2) *If  $H_{\text{cris}}^n(X/W)$  is torsion-free, and if the Frölicher spectral sequence of  $X$  degenerates at  $E_1$ , then for all  $t \in [0, \text{rank } H^n]$ , we have*

$$\text{Hdg}_{H^n}(t) = \widetilde{\text{Hdg}}_X^n(t).$$

*In particular,  $H^n$  computes all Hodge numbers  $h^{i, n-i}$  of  $X$ .*

The following exercise shows that there are restrictions on the slopes of  $F$ -crystals that arise as crystalline cohomology of varieties.

**Exercise 3.9.** Let  $X$  be smooth and proper variety of dimension  $d$ , and let  $(H^n, \varphi)$ ,  $n = 0, \dots, 2d$  be the  $F$ -crystals  $H_{\text{cris}}^n(X/W)/\text{torsion}$  as above.

- (1) Using Poincaré duality, show that  $\varphi \circ \varphi^\vee = p^d \cdot \text{id}$  for all  $n \geq 0$ , and deduce that the slopes of  $H^n$  lie inside the interval  $[0, d]$ .
- (2) Use the hard Lefschetz theorem together with Poincaré duality to show that the slopes of  $H^n$  lie inside the interval

$$\begin{aligned} & [0, n] && \text{if } 0 \leq n \leq d \\ & [n-d, d] && \text{if } d \leq n \leq 2d. \end{aligned}$$

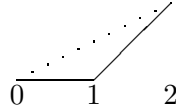
We refer to [Ka79] for more about crystals and their slopes.

**3.4. Abelian varieties.** Let  $A$  be an Abelian variety of dimension  $g$  over an algebraically closed field  $k$  of positive characteristic. As already mentioned in Examples 1.8, there exist isomorphisms

$$H_{\text{cris}}^n(A/W) \cong \Lambda^n H_{\text{cris}}^1(A/W) \quad \text{for all } n \geq 0.$$

In fact, these isomorphisms are compatible with Frobenius actions on both sides, and thus, are isomorphisms of  $F$ -crystals. In particular, it suffices to understand the  $F$ -crystal  $H_{\text{cris}}^1(A/W)$ , which is a free  $W$ -module of rank  $2g$ . Thus,  $H_{\text{cris}}^*(A/W)$  is torsion-free and since the Frölicher spectral sequence of  $A$  degenerates at  $E_1$  (see Section 1.2), the assumptions of Theorem 3.8 are fulfilled. Let us now discuss the two cases  $g = 1$  and  $g = 2$  in greater detail.

**3.4.1. Elliptic curves.** If  $A$  is an elliptic curve, that is,  $g = 1$ , then its Hodge polygon is given by the solid polygon

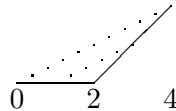


For the Newton polygon, there are two possibilities:

- (1) The Newton polygon is equal to the Hodge polygon, and in this case,  $A$  is called *ordinary*. It follows from the results of Examples 1.8 that  $A$  is ordinary if and only if  $A[p](k) \cong \mathbb{Z}/p\mathbb{Z}$ .
- (2) The Newton polygon is equal to the dotted line, and in this case,  $A$  is called *supersingular*. This case is equivalent to  $A[p](k) = \{0\}$ .

By a result of Deuring, there are roughly  $p/12$  supersingular elliptic curves over an algebraically closed field of positive characteristic  $p$ , whereas all the other ones are ordinary (see Theorem 8.3 for a similar count for K3 surfaces). In particular, a generic elliptic curve in positive characteristic is ordinary, that is, Newton and Hodge polygon of  $H_{\text{cris}}^1(A/W)$  coincide. We refer the interested reader to [Har77, Chapter IV.4] and [Si86, Chapter V] for more results, reformulations, and background information on ordinary and supersingular elliptic curves.

**3.4.2. Abelian surfaces.** If  $A$  is an Abelian surface, that is,  $g = 2$ , then its Hodge polygon is given by the solid polygon



For the Newton polygon, there are now three possibilities:

- (1) The Newton polygon is equal to the Hodge polygon, that is,  $A$  is ordinary, or, equivalently,  $A[p](k) \cong (\mathbb{Z}/p\mathbb{Z})^2$ .
- (2) The Newton polygon has three slopes (lower dotted line), or, equivalently,  $A[p](k) \cong \mathbb{Z}/p\mathbb{Z}$ .
- (3) The Newton polygon has only one slope (upper dotted line), that is,  $A$  is supersingular, or, equivalently,  $A[p](k) = \{0\}$ .

We refer to [Ill79a, Examples II.7.1] for details and further results.

**3.5. K3 surfaces.** Let  $X$  be a K3 surface over  $k$ . In Section 2.2, we computed the cohomology groups of a K3 surface. In particular, the only interesting crystalline cohomology group is  $H_{\text{cris}}^2(X/W)$ , which is free of rank 22. Moreover, in loc. cit. we also computed the Hodge numbers

$$\tilde{h}_0 := h^{0,2} = 1, \quad \tilde{h}_1 := h^{1,1} = 20, \quad \text{and} \quad \tilde{h}_2 := h^{2,0} = 1$$

from which we obtain the geometric Hodge polygon. In Section 2.2, we have also seen that the crystalline cohomology groups of  $X$  have no  $p$ -torsion, and that the Frölicher spectral sequence degenerates at  $E_1$ . Thus, by Theorem 3.8, the geometric Hodge polygon of  $X$  coincides with the Hodge polygon of the  $F$ -crystal  $H_{\text{cris}}^2(X/W)$ .

**Exercise 3.10.** For a K3 surface  $X$ , show that there are 12 possibilities for the Newton polygon of the  $F$ -crystal  $H_{\text{cris}}^2(X/W)$ :

- (1) The Newton polygon has three slopes and multiplicities as follows:

$$\begin{array}{ccc} \text{slope} & 1 - \frac{1}{h} & 1 & 1 + \frac{1}{h} \\ \text{multiplicity} & h & 22 - h & h \end{array}$$

where  $h$  is an integer with  $1 \leq h \leq 11$ . In case  $h = 1$ , Hodge and Newton polygon coincide, and then,  $X$  is called *ordinary*.

- (2) The Newton polygon is of slope 1 only (upper dotted line), and then,  $X$  is called *supersingular*. In this case we set  $h = \infty$ .

A discussion and details can be found in [III79a, Section II.7.2].

Since  $X$  is projective, there exists an ample line bundle  $\mathcal{L} \in \text{Pic}(X)$  and we will see in Section 4.2 that the  $W$ -module generated by the Chern class  $c_1(\mathcal{L})$  inside  $H_{\text{cris}}^2(X/W)$  gives rise to an  $F$ -crystal of slope 1 (see also Exercise 6.18). In particular, this shows that the case  $h = 11$  in (1) of Exercise 3.10 cannot occur.

In Section 6.2, we will define the formal Brauer group of a K3 surface. In Proposition 6.17, we will see that the parameter  $h$  from Exercise 3.10 can be interpreted as the height of the formal Brauer group. And eventually, in Section 8, we will see that  $h$  gives rise to a stratification of the moduli spaces  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  from Section 2.4, and that this stratification can also be interpreted in terms of  $F$ -zips (see Definition 1.3).

#### 4. SUPERSINGULAR K3 SURFACES

For a complex K3 surface  $X$ , the deRham cohomology group  $H_{\text{dR}}^2(X/\mathbb{C})$  comes with an integral and a real structure, as well as two filtrations (Section 1.1). Moreover, Poincaré duality equips  $H_{\text{dR}}^2(X/\mathbb{C})$  with a non-degenerate bilinear form. This linear algebra data is captured in the notion of a *polarized Hodge structure of weight 2*, and such data is parametrized by their *period domain*, which is an open subset (with respect to the analytic topology) inside some Zariski-closed set of some Grassmannian.

In the previous section, we associated to a K3 surface  $X$  over a perfect field  $k$  of positive characteristic the  $F$ -crystal  $(H, \varphi)$  arising from  $H_{\text{cris}}^2(X/W)$ . This is a module over  $W = W(k)$ , which may be thought of as an integral structure.

By Theorems 1.9 and 3.8, the two Hodge filtrations of  $H_{\text{dR}}^2(X/k) \cong H/pH$  are encoded in  $(H, \varphi)$ . Poincaré duality, which also exists for crystalline cohomology (Section 1.5), equips  $(H, \varphi)$  with a non-degenerate bilinear form. This resulting structure  $(H, \varphi, \langle -, - \rangle)$  is called a *K3 crystal*, and should be thought of as the characteristic- $p$  version of a polarized Hodge structure of weight 2 arising from a K3 surface.

Following Ogus [Og79], we will only construct a period domain for *supersingular* K3 crystals. One crucial technical point is that this will be a Zariski closed subset of some Grassmannian and thus, projective over  $k$ . In general, period domains for non-supersingular K3 crystals are expected to be open subsets of Zariski-closed sets of Grassmannians, but where open is in the sense of Tate's rigid analytic spaces or Berkovich spaces, which is why these period domains are not expected to exist as schemes.

**4.1. K3 crystals.** We start by introducing K3 crystals and their Tate modules, and shortly digress on the Tate conjecture.

**Definition 4.1** (Ogus). Let  $k$  be a perfect field of positive characteristic  $p$  and let  $W = W(k)$  be its Witt ring. A *K3 crystal of rank  $n$*  over  $k$  is a free  $W$ -module  $H$  of rank  $n$  together with a  $\sigma$ -linear injective map  $\varphi : H \rightarrow H$  (that is,  $(H, \varphi)$  is an  $F$ -crystal), and a symmetric bilinear form

$$\langle -, - \rangle : H \otimes_W H \rightarrow W$$

such that

- (1)  $p^2 H \subseteq \text{im}(\varphi)$ ,
- (2)  $\varphi \otimes_W k$  has rank 1,
- (3)  $\langle -, - \rangle$  is a perfect pairing,
- (4)  $\langle \varphi(x), \varphi(y) \rangle = p^2 \sigma \langle x, y \rangle$ .

The K3 crystal is called *supersingular*, if moreover

- (5) the  $F$ -crystal  $(H, \varphi)$  is purely of slope 1.

**Example 4.2.** Let  $X$  be a K3 surface over  $k$ . By Example 3.2,  $H := H_{\text{cris}}^2(X/W)$  with Frobenius  $\varphi$  is an  $F$ -crystal. By the results of Section 2.2, it is of rank 22.

- (1) By Exercise 3.9, all slopes of  $H$  lie in the interval  $[0, 2]$ , which can also be seen from the detailed classification in Exercise 3.10. This implies that condition (1) of Definition 4.1 is fulfilled.
- (2) Poincaré duality equips  $H$  with a symmetric bilinear pairing  $\langle -, - \rangle$ , which satisfies conditions (3) and (4) of Definition 4.1 by general properties of Poincaré duality.
- (3) Since  $X$  is a K3 surface, we have  $\tilde{h}_0 := h^2(\mathcal{O}_X) = 1$ . By Theorem 3.8, we have  $h_0 = 1$  for the Hodge polygon of  $H$ . This implies that condition (2) of Definition 4.1 holds true.

Thus,  $(H, \varphi, \langle -, - \rangle)$  is a K3 crystal of rank 22. It is supersingular if and only if its Newton polygon is a straight line of slope one, which corresponds to case (2) in Exercise 3.10, that is,  $h = \infty$ .

**Exercise 4.3.** Let  $A$  be Abelian surface, that is, an Abelian variety of dimension 2. Show that Frobenius and Poincaré duality turn  $H_{\text{cris}}^2(A/W)$  into a K3 crystal of rank 6. We refer the interested reader to [Og79, Section 6], where crystals arising from (supersingular) Abelian varieties are discussed in general. For Abelian surfaces, it turns out that these crystals are closely related to K3 crystals of rank 6, see [Og79, Proposition 6.9].

**4.2. The Tate module.** If  $X$  is a smooth and proper variety over  $k$ , then there exists a crystalline Chern class map

$$c_1 : \text{Pic}(X) \rightarrow H_{\text{cris}}^2(X/W).$$

Being a homomorphism of Abelian groups,  $c_1$  satisfies for all  $\mathcal{L} \in \text{Pic}(X)$

$$c_1(F^*(\mathcal{L})) = c_1(\mathcal{L}^{\otimes p}) = pc_1(\mathcal{L}),$$

where  $F : X \rightarrow X$  denotes the absolute Frobenius morphism. In particular,  $c_1(\text{Pic}(X))$  is contained in the Abelian subgroup (in fact,  $\mathbb{Z}_p$ -submodule) of the  $F$ -crystal  $(H_{\text{cris}}^2(X/W), \varphi)$  of those elements  $x$  that satisfy  $\varphi(x) = px$ . This observation motivates the following definition.

**Definition 4.4.** Let  $(H, \varphi, \langle -, - \rangle)$  be a K3 crystal. Then, the *Tate module* of  $H$  is defined to be the  $\mathbb{Z}_p$ -module

$$T_H := \{ x \in H \mid \varphi(x) = px \}.$$

Thus, by our computation above, we have  $c_1(\text{NS}(X)) \subseteq T_H$ , and it is natural to ask whether this inclusion is in fact an equality, or, at least up to  $p$ -torsion. If  $X$  is defined over a finite field, this is the content of the *Tate conjecture*.

**Conjecture 4.5** (Tate [Ta65]). Let  $X$  be a smooth and proper surface over a finite field  $\mathbb{F}_q$  of characteristic  $p$ . Then, the following statements hold true:

- (1) The first Chern class induces an isomorphism

$$c_1 : \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p \xrightarrow{\cong} T_H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

- (2) For every prime  $\ell \neq p$ , the first Chern class induces an isomorphism

$$c_1 : \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell} \xrightarrow{\cong} H_{\text{ét}}^2(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_{\ell}(1))^{\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)},$$

where the right hand side denotes invariants under the Galois action.

- (3) The rank of  $\text{NS}(X)$  is the pole order of the zeta function  $Z(X/\mathbb{F}_q, T)$  at  $T = q^{-1}$ .

The equivalences of (1), (2), and (3) follow from the Weil conjectures, more precisely, from the Riemann hypothesis, which relates the zeta function to  $\ell$ -adic and crystalline cohomology, see [Har77, Appendix C] and [Li13, Section 9.10]. In [Ta66, Theorem 4], Tate proved this conjecture for Abelian varieties, as well as for products of curves. For K3 surfaces, it was established in several steps depending on the slopes of the  $F$ -crystal  $H_{\text{cris}}^2$  - in terms of the notations of Exercise 3.10: for  $h = 1$  by Nygaard [Ny83a], for  $h < \infty$  by Ogus and Nygaard [N-O85], and in general by Charles [Ch12], Madapusi Pera [MP13], and Maulik [Ma12].

**Theorem 4.6** (Nygaard, Nygaard–Ogus, Charles, Madapusi–Pera, Maulik). *Tate’s conjecture holds for K3 surfaces over finite fields of odd characteristic.*

Let us mention the following, somewhat curious corollary: namely, Swinnerton-Dyer observed (see [Ar74a]) that Tate’s conjecture for K3 surfaces implies that the Néron–Severi rank of a K3 surface over  $\overline{\mathbb{F}}_p$  is *even*. This was used in [B-H-T11] and [L-L12] to show that every K3 surface of *odd* Néron–Severi rank contains infinitely many rational curves, and we refer to [Li13] and [Be14] for an overview.

**4.3. Supersingular K3 surfaces.** Let us now discuss supersingular K3 crystals in greater detail, that is, K3 crystals that are of slope 1 only. It turns out that they are largely determined by their Tate-modules. In case a supersingular K3 crystal arises as  $H_{\text{cris}}^2$  of a K3 surface, the Tate conjecture predicts that the surface has Picard rank 22, that is, the K3 surface is *Shioda-supersingular*.

Firsts, let us recall a couple of facts on quadratic forms and their classification, and we refer to [Se70, Chapitre IV] for details and proofs: let  $R$  be a ring and  $\Lambda$  a free  $R$ -module of finite rank together with a symmetric bilinear form

$$\langle -, - \rangle : \Lambda \otimes_R \Lambda \rightarrow \Lambda.$$

We choose a basis  $\{e_1, \dots, e_n\}$  of  $\Lambda$ , form the matrix  $G := (g_{ij} := \langle e_i, e_j \rangle)_{i,j}$ , and define its *discriminant* to be  $\det(G)$ . A different choice of basis of  $\Lambda$  changes it by an element of  $R^{\times 2}$ , and thus, the class  $d(\Lambda)$  of  $\det(G)$  in  $R/(R^{\times 2})$  does not depend on the choice of basis. The discriminant is zero if and only if the form is *degenerate*, that is, if there exists a  $0 \neq v \in \Lambda$  such that  $\langle v, w \rangle = 0$  for all  $w \in \Lambda$ . Next, we let  $\Lambda^\vee := \text{Hom}_R(\Lambda, R)$  be the dual  $R$ -module. Via  $v \mapsto \langle v, - \rangle$ , we obtain a natural map  $\Lambda \rightarrow \Lambda^\vee$ , which is injective if and only if the form is non-degenerate. In case this map is an isomorphism, which is the case if and only if the discriminant is a unit, the form is called *perfect*.

Let us now assume that  $R$  is a DVR, say, with valuation  $\nu$ . The example we have in mind is the ring of  $p$ -adic integers  $\mathbb{Z}_p$  or, more generally, the ring  $W(k)$  of Witt vectors of a perfect field  $k$  of positive characteristic  $p$ , together with its  $p$ -adic valuation  $\text{ord}_p$ . Then, since units have valuation zero,

$$\text{ord}_p(\Lambda) := \nu(d(\Lambda))$$

is a well-defined integer, and the form is perfect if and only if  $\text{ord}_p(\Lambda) = 0$ .

Finally, we note that quadratic forms over  $\mathbb{Q}_p$  are classified by their rank, their discriminant, and their so-called Hasse invariant, see [Se70, Chapitre IV] for proofs and details. These results are the key to the following classification of Tate modules of supersingular K3 crystals from [Og79].

**Proposition 4.7** (Ogus). *Let  $(H, \varphi, \langle -, - \rangle)$  be a supersingular K3 crystal and let  $T_H$  be its Tate module. Then,*

$$\text{rank}_W H = \text{rank}_{\mathbb{Z}_p} T_H$$

*and the bilinear form  $(H, \langle -, - \rangle)$  restricted to  $T_H$  induces a non-degenerate form*

$$T_H \otimes_{\mathbb{Z}_p} T_H \rightarrow \mathbb{Z}_p,$$

*which is not perfect. More precisely,*

- (1)  $\text{ord}_p(T_H) = 2\sigma_0 > 0$  for some integer  $\sigma_0$ , called the Artin invariant.
- (2)  $(T_H, \langle -, - \rangle)$  is determined up to isometry by  $\sigma_0$ .
- (3)  $\text{rank}_W H \geq 2\sigma_0$ .
- (4) There exists an orthogonal decomposition

$$(T_H, \langle -, - \rangle) \cong (T_0, p\langle -, - \rangle) \perp (T_1, \langle -, - \rangle)$$

where  $T_0$  and  $T_1$  are  $\mathbb{Z}_p$ -lattices, whose bilinear forms are perfect, and of ranks  $\text{rank } T_0 = 2\sigma_0$  and  $\text{rank } T_1 = \text{rank}_W H - 2\sigma_0$ .

Combining this proposition with the Tate conjecture (Theorem 4.6), we obtain a characterization of those K3 surfaces whose associated K3 crystal is supersingular. Namely, let us recall from Section 2.2 that the second crystalline cohomology group of a K3 surface is a free  $W$ -module of rank 22. Using that the first crystalline Chern map is injective, this shows that the rank of the Néron–Severi group of a K3 surface can be at most 22. This said, we have the following result.

**Theorem 4.8.** *Let  $X$  be a K3 surface over an algebraically closed field of odd characteristic. Then, the following are equivalent*

- (1) The K3 crystal  $H_{\text{cris}}^2(X/W)$  is supersingular.
- (2) The Néron–Severi group  $\text{NS}(X)$  has rank 22.

**PROOF.** If  $\text{NS}(X)$  has rank 22, then  $c_1(\text{NS}(X)) \otimes_{\mathbb{Z}} W$  is a sub- $F$ -crystal of  $H_{\text{cris}}^2(X/W)$  of slope 1, thereby establishing (2)  $\Rightarrow$  (1). Conversely, assume that  $H := H_{\text{cris}}^2(X/W)$  is a supersingular  $F$ -crystal. By Proposition 4.7, the Tate module of  $H$  has rank 22. If  $X$  can be defined over a finite field of odd characteristic then Theorem 4.6 implies that  $\text{NS}(X)$  is of rank 22. If  $X$  is not definable over a finite field, then there exists some variety  $B$  over some finite field  $\mathbb{F}_q$ , such that  $X$  is definable over the function field of  $B$ . Spreading out  $X$  over  $B$  and passing to an open and dense subset of  $B$  if necessary, we may assume that  $X$  is the generic fiber of a smooth and projective family  $\mathcal{X} \rightarrow B$  of K3 surfaces over  $\mathbb{F}_q$ . Since  $H$  is supersingular, all fibers in this family also have supersingular  $H_{\text{cris}}^2$  by [Ar74a, Section 1], and since the Néron–Severi rank is constant in families of K3 surfaces with supersingular  $H_{\text{cris}}^2$  by [Ar74a, Theorem 1.1], this establishes the converse direction (1)  $\Rightarrow$  (2).  $\square$

**Remark 4.9.** K3 surfaces satisfying (1) are called *Artin-supersingular*, see [Ar74a], where it is formulated in terms of formal Brauer groups, a point of view that we will discuss in Section 6 below. K3 surfaces satisfying (2) are called *Shioda-supersingular*, see [Sh74a]. In view of the theorem, a K3 surface in odd characteristic satisfying (1) or (2) is simply called *supersingular*.

**Examples 4.10.** Let us give examples of supersingular K3 surfaces.

- (1) Let  $A$  be a supersingular Abelian surface in odd characteristic (see Section 3.4). Then, the Kummer surface  $X := \text{Km}(A)$  of  $A$  is a supersingular K3 surface. Let  $\sigma_0$  be the Artin invariant of the Tate module  $T_H$  of the

supersingular K3 crystal  $H := H_{\text{cris}}^2(X/W)$ . Then,

$$\sigma_0(T_H) = \begin{cases} 1 & \text{if } A = E \times E, \text{ where } E \text{ is a} \\ & \text{supersingular elliptic curve, and} \\ 2 & \text{else,} \end{cases}$$

see [Og79, Theorem 7.1 and Corollary 7.14], or [Sh79, Proposition 3.7 and Theorem 4.3]. Conversely, by loc. cit., every supersingular K3 surface in odd characteristic with  $\sigma_0 \leq 2$  is the Kummer surface of a supersingular Abelian surface.

(2) The Fermat quartic

$$X_4 := \{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\} \subset \mathbb{P}_k^3$$

defines a K3 surface in characteristic  $p \neq 2$  and it is supersingular if and only if  $p \equiv 3 \pmod{4}$  by [Sh74a, Corollary to Proposition 1]. Moreover, if  $X_4$  is supersingular, then it has  $\sigma_0(T_H) = 1$  by [Sh79, Example 5.2], and thus, it is a Kummer surface by the previous example.

We note that supersingular Kummer surfaces form a 1-dimensional family, whereas all supersingular K3 surfaces form a 9-dimensional family - we refer to Section 5 for moduli spaces.

In view of Theorem 4.8, we now identify the Néron–Severi lattices arising from supersingular K3 surfaces abstractly, and classify them in terms of discriminants, which gives rise to the Artin invariant of such a lattice.

**Definition 4.11.** A *supersingular K3 lattice* is a free Abelian group  $N$  of rank 22 with an even symmetric bilinear form  $\langle -, - \rangle$  with the following properties

- (1) The discriminant  $d(N \otimes_{\mathbb{Z}} \mathbb{Q})$  is  $-1$  in  $\mathbb{Q}^*/\mathbb{Q}^{*2}$ .
- (2) The signature of  $(N \otimes_{\mathbb{Z}} \mathbb{R})$  is  $(1, 21)$ .
- (3) The cokernel of  $N \rightarrow N^{\vee}$  is annihilated by  $p$ .

We note that we follow [Og83, Definition 1.6], which is slightly different from [Og79, Definition 3.17] (in the latter article, it is stated for  $\mathbb{Z}_p$ -modules rather than  $\mathbb{Z}$ -modules). Let us shortly collect the facts: by [Og83, (1.6)] and the references given there, the Néron–Severi lattice of a supersingular K3 surface is a supersingular K3 lattice in the sense of Definition 4.11 (for example, condition (2) follows from the Hodge index theorem). Next, if  $N$  is a supersingular K3 lattice, then its discriminant  $d(N)$ , which is an integer, is equal to  $-p^{2\sigma_0}$  for some integer  $1 \leq \sigma_0 \leq 10$ .

**Definition 4.12.** The integer  $\sigma_0$  associated to a supersingular K3 lattice is called the *Artin invariant* of the lattice. If  $X$  is a supersingular K3 surface, we define its Artin invariant to be the Artin invariant of its Néron–Severi lattice.

This invariant was introduced in [Ar74a], and an important result is the following theorem, see [R-S76, Section 1] and [Og79, Section 3].

**Theorem 4.13** (Rudakov–Shafarevich). *The Artin invariant determines a supersingular K3 lattice up to isometry.*



We refer the interested reader to [R-S78, Section 1] for explicit descriptions of these lattices, which do exist for all values  $1 \leq \sigma_0 \leq 10$ .

Before proceeding, let us shortly digress on quadratic forms over finite fields: let  $V$  be a  $2n$ -dimensional vector space over a finite field  $\mathbb{F}_q$  of odd characteristic. Let  $\langle -, - \rangle : V \times V \rightarrow \mathbb{F}_q$  be a non-degenerate quadratic form. Two-dimensional examples are the hyperbolic plane  $U$ , as well as  $\mathbb{F}_{q^2}$  with the quadratic form arising from the norm. By the classification of quadratic forms over finite fields,  $V$  is isometric to  $nU$  or to  $(n-1)U \perp \mathbb{F}_{q^2}$ . The form  $\langle -, - \rangle$  is called *non-neutral* if there exists no  $n$ -dimensional isotropic subspace inside  $V$ . By the classification result just mentioned, there is precisely one non-neutral quadratic space of dimension  $2n$  over  $\mathbb{F}_q$ , namely,  $(n-1)U \perp \mathbb{F}_{q^2}$ .

Next, for a supersingular K3 lattice  $(N, \langle -, - \rangle)$ , we set  $N_1 := N/pN^\vee$ . Then,  $N_1$  is a  $(22 - 2\sigma_0)$ -dimensional  $\mathbb{F}_p$ -vector space and  $\langle -, - \rangle$  induces a quadratic form on  $N_1$ , which is non-degenerate and non-neutral. The form  $\langle -, - \rangle$  on  $pN^\vee \subseteq N$  is divisible by  $p$  and dividing it by  $p$  we obtain a non-degenerate and non-neutral bilinear form on the  $2\sigma_0$ -dimensional  $\mathbb{F}_p$ -vector space  $N_0 := pN^\vee/pN$ . We refer to [Og83, (1.6)] for details. In Section 4.4 below, we will use these  $\mathbb{F}_p$ -vector spaces to classify supersingular K3 crystals explicitly as well as to construct their moduli spaces - the point is that it is easier to deal with  $\mathbb{F}_p$ -vector spaces rather than  $\mathbb{Z}$ - or  $\mathbb{Z}_p$ -lattices.

Finally, for a supersingular K3 lattice  $N$ , we set  $\Gamma := N \otimes_{\mathbb{Z}} \mathbb{Z}_p$  and denote the induced bilinear form on  $\Gamma$  again by  $\langle -, - \rangle$ . Then, we have  $\text{ord}_p(\Gamma) = 2\sigma_0$ . By [Og79, Lemma 3.15], non-neutrality of the form induced on  $N_0$  is equivalent to the Hasse invariant of  $\Gamma$  being equal to  $-1$ . Moreover, since the cokernel of  $N \rightarrow N^\vee$  is annihilated by  $p$ , the same is true for  $\Gamma \rightarrow \Gamma^\vee$ , and thus, by [Og79, Lemma 3.14], we obtain an orthogonal decomposition

$$(\Gamma, \langle -, - \rangle) \cong (\Gamma_0, p\langle -, - \rangle) \perp (\Gamma_1, \langle -, - \rangle),$$

where  $\Gamma_0$  and  $\Gamma_1$  are perfect  $\mathbb{Z}_p$ -lattices of ranks  $2\sigma_0$  and  $22 - 2\sigma_0$ , respectively. In particular,  $\Gamma$  satisfies the conditions of a supersingular K3 lattice over  $\mathbb{Z}_p$  as defined in [Og79, Definition 3.17]. We refer to [Og79, Corollary 3.18] for details about the classification of supersingular K3 lattices over  $\mathbb{Z}_p$  up to isogeny and up to isomorphism.

**4.4. Characteristic subspaces.** In order to classify supersingular K3 crystals, we now describe them in terms of so-called characteristic subspaces, and then, classify these latter ones. For a supersingular K3 surface, this characteristic subspace arises from the kernel of the deRham Chern class  $c_1 : \text{NS}(X) \rightarrow H_{\text{dR}}^2(X/k)$ . (Note that in characteristic zero,  $c_1$  is injective modulo torsion.) These considerations stress yet again the close relation between crystals and deRham cohomology.

**Definition 4.14.** Let  $\sigma_0 \geq 1$  be an integer, let  $V$  be a  $2\sigma_0$ -dimensional  $\mathbb{F}_p$ -vector space with  $p \neq 2$ , and let

$$\langle -, - \rangle : V \times V \rightarrow \mathbb{F}_p$$

be a non-degenerate and non-neutral quadratic form. Next, let  $k$  be a perfect field of characteristic  $p$  and set  $\varphi := \text{id}_V \otimes F_k : V \otimes_{\mathbb{F}_p} k \rightarrow V \otimes_{\mathbb{F}_p} k$ . A subspace  $K \subset V \otimes_{\mathbb{F}_p} k$  is called *characteristic* if

- (1)  $K$  is totally isotropic of dimension  $\sigma_0$ , and
- (2)  $K + \varphi(K)$  is of dimension  $\sigma_0 + 1$ .

Moreover, a characteristic subspace  $K$  is *strictly characteristic* if moreover

- (3)

$$V \otimes_{\mathbb{F}_p} k = \sum_{i=0}^{\infty} \varphi^i(K)$$

holds true.

For a perfect field  $k$  of odd characteristic, we define the categories

$$\text{K3}(k) := \left\{ \begin{array}{l} \text{category of supersingular K3 crystals} \\ \text{with only isomorphisms as morphisms} \end{array} \right\}.$$

and

$$\mathbb{C}3(k) := \left\{ \begin{array}{l} \text{category of pairs } (T, K), \text{ where } T \text{ is a supersingular} \\ \text{K3 lattice over } \mathbb{Z}_p, \text{ and where } K \subset T_0 \otimes_{\mathbb{Z}_p} k \text{ is a} \\ \text{strictly characteristic subspace,} \\ \text{with only isomorphisms as morphisms} \end{array} \right\}.$$

Finally, we define  $\mathbb{C}3(k)_{\sigma_0}$  to be the subcategory of  $\mathbb{C}3(k)$ , whose characteristic subspaces are  $\sigma_0$ -dimensional.

**Theorem 4.15** (Ogus). *Let  $k$  be an algebraically closed field of odd characteristic. Then, the assignment*

$$\begin{array}{ccc} \text{K3}(k) & \xrightarrow{\gamma} & \mathbb{C}3(k) \\ (H, \varphi, \langle -, - \rangle) & \mapsto & (T_H, \ker(T_H \otimes_{\mathbb{Z}_p} k \rightarrow H \otimes_{\mathbb{Z}_p} k) \subset T_0 \otimes_{\mathbb{Z}_p} k) \end{array}$$

*defines an equivalence of categories.*

The map  $\gamma$  from the theorem has the following geometric origin, and we refer to [Og83, Section 2] for details: if  $X$  is a supersingular K3 surface, then  $H := H_{\text{cris}}^2(X/W)$  is a supersingular K3 crystal. Moreover, the Tate module  $T_H$  is a supersingular K3 lattice over  $\mathbb{Z}_p$ , the first Chern class  $c_1$  identifies  $\text{NS}(X) \otimes_{\mathbb{Z}} W$  with  $T_H$ , and the characteristic subspace associated to  $H$  arises from the kernel of  $c_1 : \text{NS}(X) \otimes_{\mathbb{Z}} k \rightarrow H_{\text{dR}}^2(X/k)$ .

Now, we describe and classify characteristic subspaces over an algebraically closed field  $k$  of odd characteristic  $p$  explicitly, and we refer to [Og79, p. 33-34] for technical details: let  $V$  be a  $2\sigma_0$ -dimensional  $\mathbb{F}_p$ -vector space with a non-neutral form  $\langle -, - \rangle$ , let  $\varphi = \text{id} \otimes F_k : V \otimes_{\mathbb{F}_p} k \rightarrow V \otimes_{\mathbb{F}_p} k$ , and let  $K \subset V \otimes_{\mathbb{F}_p} k$  be a strictly characteristic subspace. Then,

$$\ell_K := K \cap \varphi(K) \cap \dots \cap \varphi^{\sigma_0-1}(K)$$

is a line inside  $V \otimes_{\mathbb{F}_p} k$ . We choose a basis element  $0 \neq e \in \ell_K$  and set

$$e_i := \varphi^{i-1}(e) \quad \text{for } i = 1, \dots, 2\sigma_0.$$

Then, the  $\{e_i\}$  form a basis of  $V \otimes_{\mathbb{F}_p} k$ . We have  $\langle e, e_{\sigma_0+1} \rangle \neq 0$ , and changing  $e$  by a scalar if necessary (here, we use that  $k$  is algebraically closed), we may assume  $\langle e, e_{\sigma_0+1} \rangle = 1$ . We note that this normalization makes  $e$  unique up to a  $(p^{\sigma_0} + 1)$ .th root of unity. Then, we define

$$a_i := a_i(e, V, K) := \langle e, e_{\sigma_0+1+i} \rangle \quad \text{for } i = 1, \dots, \sigma_0 - 1.$$

If  $\zeta$  is a  $(p^{\sigma_0} + 1)$ .th root of unity, then, replacing  $e$  by  $\zeta e$ , transforms the  $a_i$  as  $a_i \mapsto \zeta^{1-p^i} a_i$ . This said, we denote by  $\mu_n$  the group scheme of  $n$ .th roots of unity, and then, we have the following classification result.

**Theorem 4.16** (Ogus). *Let  $k$  be an algebraically closed field of odd characteristic. Then, there exists a bijection*

$$\begin{array}{ccc} \mathbb{C}3(k)_{\sigma_0} & \rightarrow & \mathbb{A}_k^{\sigma_0-1}(k)/\mu_{p^{\sigma_0}+1}(k) \\ K & \mapsto & (a_1, \dots, a_{\sigma_0-1}) \end{array}$$

where the  $a_i := a_i(e, V, K)$  are as defined above.

Having described characteristic subspaces over algebraically closed fields, we now study them in families.

**Definition 4.17.** Let  $(V, \langle -, - \rangle)$  be a  $2\sigma_0$ -dimensional  $\mathbb{F}_p$ -vector space with a non-neutral quadratic form. If  $A$  is an  $\mathbb{F}_p$ -algebra, a *geneatrix* of  $V \otimes_{\mathbb{F}_p} A$  is a direct summand

$$K \subset V \otimes_{\mathbb{F}_p} A$$

of rank  $\sigma_0$  such that  $\langle -, - \rangle$  restricted to  $K$  vanishes identically. We define the set of geneatrices

$$\underline{\text{Gen}}_V(A) := \{ \text{geneatrices of } V \otimes_{\mathbb{F}_p} A \}$$

as well as

$\underline{M}_V(A) := \{ K \in \underline{\text{Gen}}_V(A), K + F_A^*(K) \text{ is a direct summand of rank } \sigma_0 + 1 \}$ , which is the set of characteristic geneatrices.

**Proposition 4.18** (Ogus). *The functor from  $\mathbb{F}_p$ -algebras to sets given by*

$$A \mapsto \underline{M}_V(A)$$

*is representable by a scheme  $M_V$ , which is smooth, projective, and of dimension  $\sigma_0 - 1$  over  $\mathbb{F}_p$ .*

Let  $N$  be a supersingular K3 lattice with Artin invariant  $\sigma_0$ . At the end of Section 4.3 we set  $N_0 := pN^\vee/pN$  and noted that it is a  $2\sigma_0$ -dimensional  $\mathbb{F}_p$ -vector space that inherits a non-degenerate and non-neutral bilinear form from  $N$ . We set  $\mathcal{M}_N := M_{N_0}$ .

**Definition 4.19.**  $\mathcal{M}_N$  is called the *moduli space of  $N$ -rigidified K3 crystals*.

**Examples 4.20.** If  $V$  is  $2\sigma_0$ -dimensional, then

- (1) If  $\sigma_0 = 1$ , then  $M_V \cong \text{Spec } \mathbb{F}_{p^2}$ .
- (2) If  $\sigma_0 = 2$ , then  $M_V \cong \mathbb{P}_{\mathbb{F}_{p^2}}^1$ .

(3) If  $\sigma_0 = 3$ , then  $M_V$  is isomorphic to a smooth quadric surface in  $\mathbb{P}_{\mathbb{F}_{p^2}}^3$ .

We refer to [Og79, Examples 4.7] for details, as well as to Theorem 7.9 for a generalization to higher dimensional  $V$ 's.

Anticipating the crystalline Torelli theorem in Section 5, let us comment on the  $\sigma_0 = 1$ -case and give a geometric interpretation: then, we have  $M_V \cong \text{Spec } \mathbb{F}_{p^2}$ . By Theorem 5.5 or Examples 4.10, there exists precisely one supersingular K3 surface with  $\sigma_0 = 1$  up to isomorphism over algebraically closed fields of odd characteristic. More precisely, this surface is the Kummer surface  $\text{Km}(E \times E)$ , where  $E$  is a supersingular elliptic curve. Although this surface can be defined over  $\mathbb{F}_p$ , there is no model  $X$  over  $\mathbb{F}_p$  such that all classes of  $\text{NS}(X_{\overline{\mathbb{F}}_p})$  are already defined over  $\mathbb{F}_p$ . Models with full Néron–Severi group do exist over  $\mathbb{F}_{p^2}$  – but then, there is a non-trivial Galois-action of  $\text{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$  on  $\text{NS}(X_{\mathbb{F}_{p^2}})$ . This explains (via the crystalline Torelli theorem) the Galois action on  $M_V$ , as well as the fact that  $M_V \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$  consists of two points, whereas it corresponds to only one surface.

## 5. OGUS' CRYSTALLINE TORELLI THEOREM

We now come to the period map and the crystalline Torelli theorem for supersingular K3 surfaces. To state it, we fix a prime  $p \geq 5$  and a supersingular K3 lattice  $N$  as in Definition 4.11. Then, there exists a moduli space  $\mathcal{S}_N$  of  $N$ -marked supersingular K3 surfaces, which is a scheme that is locally of finite type, almost proper, and smooth of dimension  $\sigma_0(N) - 1$  over  $\mathbb{F}_p$ .

Associating to an  $N$ -marked supersingular K3 surface  $X$  the  $F$ -zip associated to  $H_{\text{dR}}^2(X/k)$  yields a morphism

$$\pi_N^{\text{mod } p} : \mathcal{S}_N \rightarrow \mathcal{F}^\tau$$

(notation as at the end of Section 1.2). However,  $\mathcal{F}^\tau$  is a rather discrete object, it essentially only remembers the Artin invariant  $\sigma_0(X)$ , and thus,  $\pi_N^{\text{mod } p}$  is more of a “mod  $p$  shadow” of the sought-after period map.

Associating to an  $N$ -marked supersingular K3 surface the  $N$ -rigidified K3 crystal associated to  $H_{\text{cris}}^2(X/W)$  yields a morphism

$$\pi_N : \mathcal{S}_N \rightarrow \mathcal{M}_N$$

(see Definition 4.19), which is locally of finite type, étale, and surjective, but it is not an isomorphism.

If we equip  $N$ -rigidified K3 crystals with ample cones, we obtain a new moduli space  $\mathcal{P}_N$  that comes with a forgetful morphism  $\mathcal{P}_N \rightarrow \mathcal{M}_N$ , and then,  $\pi_N$  lifts to a morphism

$$\tilde{\pi}_N : \mathcal{S}_N \rightarrow \mathcal{P}_N,$$

the *period map*. By *Ogus' crystalline Torelli theorem* for supersingular K3 surfaces, the period map is an isomorphism.

**5.1. Moduli of marked supersingular K3 surfaces.** Let  $N$  be a supersingular K3 lattice in characteristic  $p$  as in Definition 4.11, and let  $\sigma_0$  be its Artin invariant as in Definition 4.12. For an algebraic space  $S$  over  $\mathbb{F}_p$ , we denote by  $\underline{N}_S$  the constant group algebraic space defined by  $N$  over  $S$ . Then, we consider the functor  $\underline{\mathcal{S}}_N$  of  $N$ -marked K3 surfaces

$$\begin{aligned} \left( \begin{array}{c} \text{Algebraic spaces} \\ \text{over } \mathbb{F}_p \end{array} \right) &\rightarrow (\text{Sets}) \\ S &\mapsto \left\{ \begin{array}{l} \text{smooth and proper morphisms } f : X \rightarrow S \\ \text{of algebraic spaces, each of whose geometric} \\ \text{fibers is a K3 surface, together with a} \\ \text{morphism of group spaces } \underline{N}_S \rightarrow \text{Pic}_{X/S} \\ \text{compatible with intersection forms.} \end{array} \right\} \end{aligned}$$

Since an  $N$ -marked K3 surface  $X \rightarrow S$  has no non-trivial automorphisms [Og83, Lemma 2.2], it is technical, yet straight forward to prove that this functor can be represented by an algebraic space [Og83, Theorem 2.7]. It follows a posteriori from the crystalline Torelli theorem [Og83, Theorem III'] that this algebraic space is a scheme.

**Theorem 5.1** (Ogus). *The functor  $\underline{\mathcal{S}}_N$  is represented by a scheme  $\mathcal{S}_N$ , which is locally of finite type, almost proper, and smooth of dimension  $\sigma_0(N) - 1$  over  $\mathbb{F}_p$ .*

Here, a scheme is called *almost proper* if satisfies the surjectivity part of the valuative criterion with DVR's as test rings. However, this moduli space is *not* separated. This non-separatedness arises from elementary transformations, which is analogous to degenerations of complex Kähler K3 surfaces. We refer to [B-R75, Section 7], [Mo83], and [Og83, page 380] for details and further discussion.

**5.2. The period map.** Let  $(H, \varphi)$  be the K3 crystal associated to  $H_{\text{cris}}^2(X/W)$  of a K3 surface  $X$  over a perfect field  $k$  of positive characteristic  $p$ . Moreover, if  $X$  is  $N$ -marked, then the inclusion  $N \rightarrow \text{NS}(X)$  composed with the first crystalline Chern map yields a map  $N \rightarrow T_H$ , where  $T_H$  denotes the Tate module of the K3 crystal. Thus, an  $N$ -marked K3 surface gives rise to an  $N$ -rigidified supersingular K3 crystal, which gives rise to a morphism of schemes

$$\pi : \mathcal{S}_N \rightarrow \mathcal{M}_N$$

(we refer to [Og79, Section 5] for families of crystals). Although this morphism is étale and surjective by [Og83, Proposition 1.16], it is not an isomorphism. In order to obtain an isomorphism (the period map), we have to enlarge  $\mathcal{M}_N$  by considering  $N$ -marked supersingular K3 crystals together with *ample cones*.

**Definition 5.2.** Let  $N$  be a supersingular K3 lattice. Then, we define its *roots* to be the set

$$\Delta_N := \{\delta \in N \mid \delta^2 = -2\}.$$

For a root  $\delta \in \Delta_N$ , we define the *reflection* in  $\delta$  to be the automorphism of  $N$  defined by

$$r_\delta : x \mapsto x + \langle x, \delta \rangle \cdot \delta \quad \text{for all } x \in N.$$

We denote by  $R_N$  the subgroup of  $\text{Aut}(N)$  generated by all  $r_\delta$ ,  $\delta \in \Delta_N$ . We denote by  $\pm R_N$  the subgroup of  $\text{Aut}(N)$  generated by  $R_N$  and  $\pm \text{id}$ . Finally, we define the set

$$V_N := \{x \in N \otimes \mathbb{R} \mid x^2 > 0 \text{ and } \langle x, \delta \rangle \neq 0 \text{ for all } \delta \in \Delta_N\} \subset N \otimes \mathbb{R}.$$

Then, the subset  $V_N \subset N \otimes \mathbb{R}$  is open, and each of its connected components meets  $N$ . A connected component of  $V_N$  is called an *ample cone*, and we denote by  $C_N$  the set of ample cones. Moreover, the group  $\pm R_N$  operates simply and transitively on  $C_N$ . We refer to [Og83, Proposition 1.10] for details and proof.

**Definition 5.3.** Let  $N$  be a supersingular K3 lattice, and let  $S$  be an algebraic space over  $\mathbb{F}_p$ . For a characteristic geneatrix  $K \in \mathcal{M}_N(S)$ , that is, a local direct factor  $K \subset \mathcal{O}_S \otimes N_0$  as in Definition 4.17, we set for each point  $s \in S$

$$\begin{aligned} \Lambda(s) &:= N_0 \cap K(s) \\ N(s) &:= \{x \in N \otimes \mathbb{Q} \mid px \in N \text{ and } \overline{px} \in \Lambda(s)\} \\ \Delta(s) &:= \{\delta \in N(s) \mid \delta^2 = -2\} \end{aligned}$$

An *ample cone* for  $K$  is an element

$$\alpha \in \prod_{s \in S} C_{N(s)}$$

such that  $\alpha(s) \subseteq \alpha(t)$  whenever  $s$  is a specialization of  $t$ .

Having introduced these definitions, we consider the functor  $\underline{\mathcal{P}}_N$

$$\begin{aligned} \left( \begin{array}{c} \text{Algebraic spaces} \\ \text{over } \mathbb{F}_p \end{array} \right) &\rightarrow \text{(Sets)} \\ S &\mapsto \left\{ \begin{array}{l} \text{characteristic spaces } K \in \mathcal{M}_N(S) \\ \text{together with ample cones} \end{array} \right\} \end{aligned}$$

There is a natural forgetful map  $\underline{\mathcal{P}}_N \rightarrow \mathcal{M}_N$ , given by forgetting the ample cones. Then, we have the following result, and refer to [Og83, Proposition 1.16] for details and proof.

**Theorem 5.4 (Ogus).** *The functor  $\underline{\mathcal{P}}_N$  is represented by a scheme  $\mathcal{P}_N$ , which is locally of finite type, almost proper, and smooth of dimension  $\sigma_0(N) - 1$  over  $\mathbb{F}_p$ . The natural map*

$$\mathcal{P}_N \rightarrow \mathcal{M}_N$$

*is étale, surjective, and locally of finite type.*

We repeat that the morphism  $\mathcal{P}_N \rightarrow \mathcal{M}_N$  is neither of finite type nor separated, whereas  $\mathcal{M}_N$  is smooth, projective and of finite type over  $\mathbb{F}_p$  by Proposition 4.18.

Now, for an algebraic space  $B$  over  $\mathbb{F}_p$  and a family  $X \rightarrow B$  of  $N$ -marked K3 surfaces, that is, an element of  $\mathcal{S}_N(B)$ , we have an associated family of  $N$ -rigidified supersingular K3 crystals, that is, an element of  $\mathcal{M}_N(B)$ . This gives rise to a morphism

$$\pi_N : \mathcal{S}_N \rightarrow \mathcal{M}_N,$$

which is surjective, but not an isomorphism. Now, for every point  $b \in B$ , there is a unique connected component of  $V_{\text{NS}(X_b) \otimes \mathbb{R}}$  that contains the classes of all ample

invertible sheaves of  $X_b$ , thereby equipping the family of K3 crystals with ample cones. This induces a morphism

$$\tilde{\pi}_N : \mathcal{S}_N \longrightarrow \mathcal{P}_N$$

that lifts  $\pi_N$  from above, and is called the *period map*. By [Og83, Theorem III], it is an isomorphism.

**Theorem 5.5** (Ogus' Crystalline Torelli Theorem). *Let  $N$  be a supersingular K3 lattice in characteristic  $p \geq 5$ . Then, the period map  $\tilde{\pi}_N$  is an isomorphism.*

Since ample cones are sometimes inconvenient to handle, and since they are also responsible for  $\mathcal{P}_N$  being neither separated nor of finite type, let us note the following useful application of the crystalline Torelli theorem: if two supersingular K3 surfaces have isomorphic K3 crystals, then they correspond via  $\tilde{\pi}_N$  to points in the same fiber of  $\mathcal{P}_N \rightarrow \mathcal{M}_N$ . In particular, the two surfaces are abstractly isomorphic, and we obtain the following result, and refer to [Og83, Theorem 1] for details.

**Corollary 5.6.** *Two supersingular K3 surfaces in characteristic  $p \geq 5$  are isomorphic if and only if their associated K3 crystals are isomorphic.*

Theorem 5.5 is the main result of [Og83]. Let us roughly sketch its proof: The existence of the period map  $\tilde{\pi}_N$  is clear. Separatedness of  $\tilde{\pi}_N$  follows from a theorem of Matsusaka and Mumford [M-M64]. Properness of  $\tilde{\pi}_N$  follows from a theorem of Rudakov and Shafarevich [R-S82] that supersingular K3 surfaces have potential good reduction, that is, given a supersingular K3 surface  $X$  over  $K := k((t))$ , there exists a finite extension  $R' \supseteq R := k[[t]]$ , say with field of fractions  $K'$ , and a smooth model of  $X \times_K K'$  over  $R'$  (this result uses that  $X$  is supersingular – in general, K3 surfaces do not have potential good reduction). Next,  $\tilde{\pi}_N$  is étale, which eventually follows from the fact that  $\pi_N : \mathcal{S}_N \rightarrow \mathcal{M}_N$  is étale [Og79, Theorem 5.6], which in turn rests on the description of its derivative [Og79, Corollary 5.4]. Finally, to prove that  $\tilde{\pi}_N$  is an isomorphism, it suffices to find one point  $\zeta \in \mathcal{P}_N$  such that  $\tilde{\pi}_N^{-1}(\zeta)$  consists of a single point – this is done by taking  $\zeta$  to be the supersingular K3 surface that is the Kummer surface for the self-product of a supersingular elliptic curve. We refer to [Og83, Section 3] for details.

## 6. FORMAL GROUP LAWS

In this section, we introduce formal group laws, which, at first sight, looks rather independent from what we studied so far. Before explaining, why this is not so, let us first give the prototype of such an object: let  $G$  be a group scheme, say, of finite type and smooth over a field  $k$ . If  $(\mathcal{O}_{G,O}, \mathfrak{m})$  is the local ring at the neutral element  $O \in G$ , then  $(\mathfrak{m}/\mathfrak{m}^2)^\vee$  yields the Lie algebra  $\mathfrak{g}$ , which captures first order infinitesimal information of  $G$  around  $O$ . Using the group structure, the formal completion of  $G$  along  $O$

$$\hat{G} := \operatorname{Spf} \varprojlim \mathcal{O}_{G,O}/\mathfrak{m}^n$$

becomes a group object in the category of formal schemes, and this *formal group law* lies somewhere between  $\mathfrak{g}$  and  $G$ . We will see that commutative formal group laws can be classified via their *Cartier–Dieudonné modules*, which are  $W(k)$ -modules that resemble  $F$ -crystals.

For example, if  $X$  is a smooth and proper variety over  $k$ , then the Picard scheme  $\mathrm{Pic}_{X/k}$  is a group scheme over  $k$ , whose formal completion along the origin is called the *formal Picard scheme*. If  $k$  is perfect of positive characteristic, then the Cartier–Dieudonné module of the formal Picard scheme determines  $H_{\mathrm{cris}}^1(X/W)$ .

As shown by Artin and Mazur [A-M77], the formal Picard group is just the  $n = 1$ -case of a whole series of formal group laws  $\Phi_{X/k}^n$  that arise from cohomological deformation functors, and we refer to Section 6.2 for precise definitions.

For us,  $\Phi_{X/k}^2$ , which is called the *formal Brauer group*, will be most relevant in the sequel: its Cartier–Dieudonné module controls the part of  $H_{\mathrm{cris}}^2(X/W)$  that is of slope less than 1. What makes this formal group law so fascinating is that despite its appearance it does in general *not* arise as formal completion of some group scheme associated to  $X$ . Later, in Section 7, we will use the formal Brauer group to construct non-trivial deformations of supersingular K3 surfaces, which ultimately proves their unirationality.

**6.1. Formal group laws.** We start with a short introduction to commutative formal group laws and their classification, and refer to [Ha78] for the general theory of formal group laws, and especially to [Zi84] for the theory of Cartier–Dieudonné modules.

**Definition 6.1.** An  $n$ -dimensional *formal group law* over a ring  $k$  consists of  $n$  power series  $\vec{F} = (F_1, \dots, F_n)$

$$F_i(x_1, \dots, x_n, y_1, \dots, y_n) \in k[[x_1, \dots, x_n, y_1, \dots, y_n]], \quad i = 1, \dots, n$$

such that for all  $i = 1, \dots, n$

$$\begin{aligned} F_i(\vec{x}, \vec{y}) &\equiv x_i + y_i \text{ modulo terms of degree } \geq 2, \text{ and} \\ F_i(\vec{x}, F_i(\vec{y}, \vec{z})) &= F_i(F_i(\vec{x}, \vec{y}), \vec{z}), \end{aligned}$$

where we use the notation  $\vec{x}$  to denote  $(x_1, \dots, x_n)$ , etc. A formal group law  $\vec{F}$  is called *commutative* if

$$F_i(\vec{x}, \vec{y}) = F_i(\vec{y}, \vec{x}) \quad \text{for all } i = 1, \dots, n.$$

A *homomorphism*  $\vec{\alpha} : \vec{F} \rightarrow \vec{G}$  from an  $n$ -dimensional formal group law  $\vec{F}$  to an  $m$ -dimensional formal group law  $\vec{G}$  consists of  $m$  formal power series  $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$  in  $n$  variables such that  $\alpha_i(\vec{x}) \equiv 0 \pmod{\text{degree } 1}$  and

$$\begin{aligned} \alpha_i(\vec{x}) &\equiv 0 \text{ modulo terms of degree } \geq 1, \text{ and} \\ \vec{\alpha}(\vec{F}(\vec{x}, \vec{y})) &= \vec{G}(\vec{\alpha}(\vec{x}), \vec{\alpha}(\vec{y})). \end{aligned}$$

It is called an *isomorphism* if there exists a homomorphism  $\vec{\beta} : \vec{G} \rightarrow \vec{F}$  such that  $\vec{\alpha}(\vec{\beta}(\vec{x})) = \vec{x}$  and  $\vec{\beta}(\vec{\alpha}(\vec{y})) = \vec{y}$ . An isomorphism  $\vec{\alpha}$  is called *strict* if  $\alpha_i(\vec{x}) \equiv x_i \pmod{\text{terms of degree } \geq 2}$  for all  $i = 1, \dots, n$ .



For example, for an integer  $n \geq 1$ , we define *multiplication by  $n$*  to be

$$[n](\vec{x}) := \underbrace{\vec{F}(\vec{F}(\dots, \vec{x}), \vec{x})}_{n \text{ times}} \in k[[\vec{x}]].$$

If  $\vec{F}$  is a commutative formal group law, then  $[n] : \vec{F} \rightarrow \vec{F}$  is a homomorphism of formal group laws. The group laws arising from algebraic varieties, which we will introduce in Section 6.2 below, will all be commutative. Let us also mention the following result, which will not need in the sequel, and refer to [Ha78, Section 6.1] for details.

**Theorem 6.2.** *A 1-dimensional formal group law over a reduced ring is commutative.*

**Examples 6.3.** Here are two basic examples of 1-dimensional formal group laws.

- (1) The *formal additive group*  $\widehat{\mathbb{G}}_a$  is defined by  $F(x, y) = x + y$ .
- (2) The *formal multiplicative group*  $\widehat{\mathbb{G}}_m$  is defined by  $F(x, y) = x + y + xy$ .

Both are commutative, and their names will be explained in Example 6.10.

Over  $\mathbb{Q}$ -algebras, all commutative formal group laws of the same dimension are mutually isomorphic:

**Theorem 6.4.** *Let  $\vec{F}$  be an  $n$ -dimensional commutative formal group law over a  $\mathbb{Q}$ -algebra. Then, there exists a unique strict isomorphism*

$$\log_{\vec{F}} : \vec{F}(\vec{x}, \vec{y}) \rightarrow \widehat{\mathbb{G}}_a^n,$$

*called the logarithm of  $\vec{F}$ .*

**Example 6.5.** The logarithm of the formal multiplicative group  $\widehat{\mathbb{G}}_m$  over a  $\mathbb{Q}$ -algebra  $k$  is explicitly given by

$$\log_{\widehat{\mathbb{G}}_m} : x \mapsto \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

which also motivates the name. Note that this power series only makes sense in rings that contain  $\frac{1}{n}$  for all integers  $n \geq 1$ , that is, the base ring must be a  $\mathbb{Q}$ -algebra.

On the other hand, if  $k$  is an  $\mathbb{F}_p$ -algebra, then Theorem 6.4 no longer holds true. In fact, there exist many 1-dimensional commutative formal group laws over algebraically closed fields of positive characteristic that are *not* isomorphic to  $\widehat{\mathbb{G}}_a$ . The following discrete invariant is crucial - we will only define it for 1-dimensional formal group laws and refer to [Ha78, (18.3.8)] for its definition for higher dimensional formal group laws.

**Definition 6.6.** Let  $F = F(x, y)$  be a 1-dimensional commutative formal group law over a field  $k$  of positive characteristic  $p$ , and let  $[p] : F \rightarrow F$  be multiplication by  $p$ . Then, the *height*  $h = h(F)$  of  $F$  is defined to be

-  $h := \infty$  in case  $[p] = 0$ , and one also says that  $F$  is *unipotent*.

- Else, there exists an integer  $s \geq 1$  and a  $0 \neq a \in k$ , such that

$$[p](x) = a \cdot x^{p^s} + \text{higher order terms},$$

in which case we set  $h := s$ . If  $F$  is of finite height, one also says that it is *p-divisible*.

**Remark 6.7.** Let us give more characterizations of the height of a 1-dimensional formal group law  $F = F(x, y)$  over a field  $k$  of positive characteristic  $p$ . To do so, we define  $F^{(p)} = F^{(p)}(x, y)$  to be the formal group law obtained from  $F$  by raising each of the coefficients to the  $p$ .th power. Then,  $z \mapsto z^p$  defines a homomorphism of formal group laws over  $k$

$$\sigma : F \rightarrow F^{(p)},$$

which is called *Frobenius*.

- (1) If finite, the height is characterized as being the largest integer  $h$  such that there exists a power series  $\beta \in k[[z]]$  with

$$[p](z) = \beta(z^{p^h}).$$

The series  $\beta$  defines a homomorphism  $F^{(p^h)} \rightarrow F$ , which leads to the following reformulation:

- (2) If finite, the height is characterized as being the largest integer  $h$  such that there exists a factorization

$$F \xrightarrow{\sigma^{oh}} F^{(p^h)} \xrightarrow{\exists} F, \\ [p]$$

where  $\sigma^{oh}$  denotes the  $h$ -fold composition of  $\sigma$  with itself.

- (3) Finally, if the height is finite, then  $[p]$  is an *isogeny*, and then, there exists an integer  $m \geq 0$  and a homomorphism of formal group laws  $\psi : F \rightarrow F^{(p^m)}$  such that  $\psi \circ [p] = \sigma^{om}$ . In particular, if finite, the height is characterized as being the smallest integer  $h$  such that there exists a factorization

$$F \xrightarrow{[p]} F \xrightarrow{\exists} F^{(p^h)}. \\ \sigma^{oh}$$

We refer to [Ha78, Section 18.3] for details and generalizations.

**Exercise 6.8.** Over fields of positive characteristic, show that

$$h(\widehat{\mathbb{G}}_m) = 1 \quad \text{and} \quad h(\widehat{\mathbb{G}}_a) = \infty.$$

In particular, they are not isomorphic.

The importance of the height lies in the following classification result.

**Theorem 6.9** (Lazard). *Let  $k$  be an algebraically closed field of positive characteristic.*

- (1) *For every integer  $h \geq 1$  or  $h = \infty$  there exists a 1-dimensional formal group law of height  $h$  over  $k$ .*

- (2) *Two 1-dimensional formal group laws over  $k$  are isomorphic if and only if they have the same height.*

**Example 6.10.** Let us show how formal group laws arise from group schemes, which also justifies some of the terminology introduced above. Thus, let  $G$  be a smooth (commutative) group scheme of dimension  $n$  over a field  $k$ . Let  $(\mathcal{O}_{G,O}, \mathfrak{m})$  be the local ring at the neutral element  $O \in G$ . Using smoothness, we compute the  $\mathfrak{m}$ -adic completion to be

$$\widehat{\mathcal{O}}_{G,O} := \varprojlim \mathcal{O}_{G,O}/\mathfrak{m}^m \cong k[[x_1, \dots, x_n]] = k[[\vec{x}]],$$

and note that

$$\widehat{G} := \mathrm{Spf} \widehat{\mathcal{O}}_{G,O}$$

is the *formal completion of  $G$  along  $O$* . The multiplication  $\mu : G \times G \rightarrow G$  induces a morphism of formal schemes  $\widehat{\mu} : \widehat{G} \times \widehat{G} \rightarrow \widehat{G}$  that turns  $\widehat{G}$  into a (commutative) group object in the category of formal schemes. Explicitly,  $\widehat{\mu}$  corresponds to a homomorphism of  $k$ -algebras

$$\widehat{\mu}^\# : k[[\vec{x}]] \rightarrow k[[\vec{y}]] \widehat{\otimes} k[[\vec{z}]] \cong k[[\vec{y}, \vec{z}]],$$

where  $\widehat{\otimes}$  denotes the completed tensor product. Clearly,  $\widehat{\mu}^\#$  is uniquely determined by the images of the generators  $x_i$ , that is, by the  $n$  formal power series

$$\vec{G} := (G_1, \dots, G_n) := \widehat{\mu}^\#(x_1, \dots, x_n).$$

It is not difficult to see that  $\vec{G}$  is an  $n$ -dimensional (commutative) formal group law, and that it encodes  $\widehat{G}$ . Thus,  $\widehat{G}$  carries the information of all infinitesimal neighborhoods of  $O$  in  $G$ , and in particular, of the tangent space at  $O$ , that is, the Lie algebra  $\mathfrak{g}$  of  $G$ . Put a little bit sloppily,  $\widehat{G}$  lies between  $G$  and  $\mathfrak{g}$ . Here are some standard examples

- (1) The completion of the multiplicative group scheme  $\mathbb{G}_m \cong \mathrm{Spec} k[x, x^{-1}]$  is the formal multiplicative group law  $\widehat{\mathbb{G}}_m$ .
- (2) The completion of the additive group scheme  $\mathbb{G}_a \cong \mathrm{Spec} k[x]$  is the formal additive group law  $\widehat{\mathbb{G}}_a$ .
- (3) If  $E$  is an elliptic curve over a field  $k$ , then the completion  $\widehat{E}$  is a commutative 1-dimensional formal group law. If  $k$  is of positive characteristic, then its height is equal to

$$h(\widehat{E}) = \begin{cases} 1 & \text{if } E \text{ is ordinary, and} \\ 2 & \text{if } E \text{ is supersingular.} \end{cases}$$

Thus, if  $k$  is algebraically closed and  $E$  is ordinary, then  $\widehat{E} \cong \widehat{\mathbb{G}}_m$ . We refer to [Si86, Chapter IV] for more about this formal group law.

In order to classify 1-dimensional commutative formal group laws over perfect fields  $k$  of positive characteristic that are not algebraically closed, or even higher dimensional commutative formal group laws over  $k$ , the height is not sufficient. To

state the general classification result, which is in terms of modules over some non-commutative ring  $\text{Cart}(k)$ , we first have to define this ring. By definition,  $\text{Cart}(k)$  is the non-commutative power series ring  $W(k)\langle\langle F, V \rangle\rangle$  modulo the relations

$$FV = p, \quad VrF = V(r), \quad Fr = \sigma(r)F, \quad rV = V\sigma(r) \quad \text{for all } r \in W(k),$$

where  $\sigma(r), V(r) \in W(k)$  denote Frobenius and Verschiebung of  $W(k)$ .

**Theorem 6.11** (Cartier–Dieudonné). *Let  $k$  be a perfect field of positive characteristic. Then, there exists a covariant equivalence of categories between*

- (1) *The category of commutative formal group laws over  $k$ .*
- (2) *The category of left  $\text{Cart}(k)$ -modules  $M$  such that*
  - (a)  *$V$  is injective,*
  - (b)  *$\cap_i V^i(M) = 0$ , that is,  $M$  is  $V$ -adically separated, and*
  - (c)  *$M/VM$  is a finite-dimensional  $k$ -vector space.*

*The left  $\text{Cart}(k)$ -module associated to a formal group  $\vec{G}$  under this equivalence is called the Dieudonné–Cartier module of  $\vec{G}$ , and is denoted  $D\vec{G}$ .*

Following [Mu69b, Section 1], let us sketch the direction (1)  $\rightarrow$  (2) of this equivalence: let  $\vec{G} = (G_1, \dots, G_n)$  be an  $n$ -dimensional commutative formal group law over  $k$ . Then,  $\vec{G}$  defines a functor

$$\begin{aligned} \Phi_{\vec{G}} : (k\text{-algebras}) &\rightarrow (\text{Abelian groups}) \\ R &\mapsto \{(x_1, \dots, x_n) \in R^n \mid \text{each } x_i \text{ nilpotent}\} \end{aligned}$$

where we define the group structure on the right by setting  $\vec{x} \oplus_{\Phi_{\vec{G}}} \vec{y} := \vec{G}(\vec{x}, \vec{y})$ . Similarly, we define the functor

$$\begin{aligned} \Phi_{\vec{W}} : (k\text{-algebras}) &\rightarrow (\text{Abelian groups}) \\ R &\mapsto \left\{ (x_0, x_1, \dots) \left| \begin{array}{l} x_i \in R, \text{ each } x_i \text{ nilpotent,} \\ \text{and almost all } x_i = 0 \end{array} \right. \right\} \end{aligned}$$

using  $\vec{W} := (W_0, W_1, \dots)$  to define the group structure, where the  $W_i$  are the Witt polynomials from Section 1.4. We note that  $\vec{W}$  is an example of an infinite dimensional formal group law. Next, we define

$$D\vec{G} := \text{Hom}_{\text{group functors}/k}(\Phi_{\vec{W}}, \Phi_{\vec{G}}).$$

Multiplication by elements of  $W(k)$  gives rise to endomorphisms of  $\Phi_{\vec{W}}$ , and one can define Frobenius and Verschiebung for  $\Phi_{\vec{W}}$ . These endomorphisms generate a subring of  $\text{Hom}(\Phi_{\vec{W}}, \Phi_{\vec{W}})$ , which is isomorphic to the ring  $\text{Cart}(k)$  defined above.

In particular, this turns  $D\vec{G}$  into a left  $\text{Cart}(k)$ -module, which turns out to satisfy the conditions in (2) of Theorem 6.11. We refer to [Mu69b, Section 1] and [Ha78, Chapter V] for details, generalizations, as well as different approaches.

**Examples 6.12.** Let  $k$  be an algebraically closed field of positive characteristic, and let  $G = G(x, y)$  be a commutative 1-dimensional formal group law over  $k$ .

- (1) If  $G$  is of finite height  $h$ , then  $DG \cong \text{Cart}(k)/(F - V^{h-1})$ , which is a free  $W(k)$ -module of rank  $h$ .

- (2) If  $G$  is of infinite height, then  $G \cong \widehat{\mathbb{G}}_a$ , and then  $D\widehat{\mathbb{G}}_a \cong k[[x]]$  with  $F = 0$  and  $V(x^n) = x^{n+1}$ .

In particular,  $h$  is equal to the minimal number of generators of  $DG$  as  $W(k)$ -module.

**Exercise 6.13.** Let  $G$  be the 1-dimensional commutative formal group law of finite height  $h$  over an algebraically closed field  $k$  of positive characteristic. Show that  $DG$ , considered as a  $W(k)$ -module with the  $\sigma$ -linear action defined by  $\varphi := F$  is an  $F$ -crystal. Show that there exists an isomorphism of  $F$ -crystals

$$DG \cong N_\alpha \quad \text{with } \alpha = (h-1)/h,$$

where  $N_\alpha$  is as in Example 3.6. In particular, it is of rank  $h$ , of slope  $\alpha = 1 - \frac{1}{h}$ , and has Hodge numbers  $h_0 = 1$ ,  $h_1 = h - 1$ , and  $h_i = 0$  if  $i \neq 0, 1$ .

**6.2. Formal groups arising from algebraic varieties.** We now explain how Artin and Mazur [A-M77] associated formal group laws to algebraic varieties: let  $X$  be a smooth and proper variety over a field  $k$ , and let  $n \geq 1$  be an integer. Then, we consider the following functor from the category  $\text{Art}_k$  of local Artinian  $k$ -algebras with residue field  $k$  to Abelian groups:

$$\begin{aligned} \Phi_X^n : \text{Art}_k &\rightarrow (\text{Abelian groups}) \\ R &\mapsto \ker \left( H_{\text{ét}}^n(X \times_k R, \mathcal{O}_{X \times_k R}^\times) \xrightarrow{\text{res}} H_{\text{ét}}^n(X, \mathcal{O}_X^\times) \right) \end{aligned}$$

where  $\mathcal{O}_X^\times$  denotes the sheaf of invertible elements of  $\mathcal{O}_X$  with respect to multiplication (we could also write  $\mathbb{G}_m$ ). The pro-representability of this functor is studied in [A-M77], and there, also a tangent-obstruction theory for it with tangent space  $H^n(X, \mathcal{O}_X)$  and obstruction space  $H^{n+1}(X, \mathcal{O}_X)$  is established.

**Example 6.14.** The case  $n = 1$  is easy to explain: Let  $R \in \text{Art}_k$ . We identify  $H_{\text{ét}}^1(X \times_k R, \mathcal{O}_{X \times_k R}^\times)$  with the group of invertible sheaves of  $X \times_k R$ , and then,  $\Phi_X^1(R)$  becomes the group of invertible sheaves on  $X \times_k R$ , whose restriction to  $X$  is trivial. Thus, elements of  $\Phi_X^1(R)$  are in bijection to morphisms  $\text{Spec } R \rightarrow \text{Pic}_{X/k}$ , such that the closed point of  $\text{Spec } R$  maps to zero, that is, the class of  $\mathcal{O}_X$ . Thus, if  $\widehat{\text{Pic}}_{X/k}$  denotes the completion of the Picard scheme  $\text{Pic}_{X/k}$  along its zero as in Example 6.10, then we obtain an isomorphism of functors

$$\widehat{\text{Pic}}_{X/k} \cong \Phi_X^1.$$

In particular,  $\Phi_X^1$  is pro-representable by a commutative formal group law if and only if  $\text{Pic}_{X/k}^0$  is smooth over  $k$ , that is, if and only if  $\text{Pic}_{X/k}^0$  is an Abelian variety. In this case, it is called the *formal Picard group*, and it is of dimension  $h^1(\mathcal{O}_X)$ .

Let us now turn to  $\Phi_X^2$ , which classifies deformations of  $0 \in H_{\text{ét}}^2(X, \mathcal{O}_X^\times)$ . The group  $H_{\text{ét}}^2(X, \mathcal{O}_X^\times)$  is called the (*cohomological*) *Brauer group* of  $X$ , and we refer to [G-S06] for its algebraic aspects, and to [Gr68a] for the more scheme-theoretic side of this group. Unlike the Picard group, there is in general no Brauer scheme, whose points parametrize elements of the Brauer group of  $X$ . In particular,  $\Phi_X^2$ ,

unlike  $\Phi_X^1$ , does *not* seem to arise as completion of some group scheme associated to  $X$ . Nevertheless, we can still study the functor  $\Phi_X^2$ . For example, if  $h^1(\mathcal{O}_X) = h^3(\mathcal{O}_X) = 0$  (this holds, for example, for K3 surfaces), then  $\Phi_X^2$  is pro-representable by a commutative formal group law of dimension  $h^2(\mathcal{O}_X)$ , the *formal Brauer group*, which is denoted

$$\widehat{\mathrm{Br}}_X := \Phi_X^2.$$

In Section 7, we will use this formal group law to construct non-trivial 1-dimensional deformations of supersingular K3 surfaces, which is the key to proving their unirationality.

For  $n \geq 3$ , the functors  $\Phi_X^n$  are far less understood. However, we refer the interested reader to [G-K03] for an analysis of  $\Phi_X^n$  in case  $X$  is an  $n$ -dimensional Calabi–Yau variety.

**6.3. The connection to Witt vector and crystalline cohomology.** In Section 1.4, we introduced Serre’s Witt vector cohomology groups  $H^n(W\mathcal{O}_X)$ , which, by functoriality of the Witt vector construction, carry actions of Frobenius and Verschiebung. In particular, all these cohomology groups are left  $\mathrm{Cart}(k)$ -modules. The following result from [A-M77, Proposition (II.2.13)] and [A-M77, Corollary (II.4.3)] links this  $\mathrm{Cart}(k)$ -module structure to the Cartier–Dieudonné modules of commutative formal group laws associated to the  $\Phi_X^n$ .

**Proposition 6.15** (Artin–Mazur). *Let  $X$  be a proper variety over a perfect field  $k$  and assume that  $\Phi_X^n$  is pro-representable by a formal group law  $\vec{F}$  (for example, this holds true if  $h^{n-1}(\mathcal{O}_X) = h^{n+1}(\mathcal{O}_X) = 0$ ). Then, there exists an isomorphism of left  $\mathrm{Cart}(k)$ -modules*

$$D\vec{F} \cong H^n(X, W\mathcal{O}_X).$$

To link the formal group law  $\Phi_X^n$  to crystalline cohomology, we use the slope spectral sequence from Section 1.5. As mentioned there, it degenerates at  $E_1$  if and only if the torsion of the Hodge–Witt cohomology groups is finitely generated. However, combining the previous proposition with Examples 6.12, we see that if  $\Phi_X^n \cong \widehat{\mathbb{G}}_a$  (for example, if  $n = 2$  and  $X$  is a supersingular K3 surface), then  $H^n(W\mathcal{O}_X)$  will not be finitely generated and the slope spectral sequence will not degenerate at  $E_1$ , see [Ill79a, Théorème II.2.3.7]. On the other hand, the slope spectral sequence always degenerates at  $E_1$  modulo torsion, and from this, we obtain an isomorphism of  $F$ -isocrystals

$$H^n(X, W\mathcal{O}_X) \otimes_W K \cong (H_{\mathrm{cris}}^n(X/W) \otimes_W K)_{[0,1[},$$

where the right hand denotes the direct sum of sub- $F$ -isocrystals of slope strictly less than 1. (Here, a little bit of background: the point is that the  $H^j(W\Omega_X^i) \otimes_W K$  are finite-dimensional  $K$  vector spaces and their sets of slopes are disjoint. The slope spectral sequence degenerates at  $E_1$  after tensoring with  $K$ , and since it can be made compatible with the Frobenius actions on both sides, the isogeny decomposition of the  $F$ -isocrystal  $H_{\mathrm{cris}}^n(X/W) \otimes_W K$  can be read off from the isogeny decomposition of the  $F$ -isocrystals  $H^j(W\Omega_X^i) \otimes_W K$ , where  $i + j = n$ .)

From this, it is not so difficult to see that all  $F$ -isocrystals of slope strictly less than 1 in  $H_{\text{cris}}^n(X/W) \otimes_W K$  arise from  $H^n(W\mathcal{O}_X) \otimes_W K$  via the slope spectral sequence.)

**Example 6.16.** Assume that  $\Phi_X^n$  is pro-representable by a 1-dimensional formal group law of *finite* height  $h$ . By Exercise 6.13 and Proposition 6.15, we have isomorphisms

$$N_{(h-1)/h} \otimes_W K \cong D\Phi_X^n \otimes_W K \cong (H_{\text{cris}}^n(X/W) \otimes_W K)_{[0,1[}.$$

In particular,  $1 - \frac{1}{h}$  is the only slope of  $H_{\text{cris}}^n(X/W)$  less than 1.

This example applies to the case where  $X$  is a K3 surface, and  $n = 2$ , that is,  $\Phi_X^n$  is the formal Brauer group. In this special case, we can say more. Before doing so, we remind the reader that we classified the possible slopes and Newton polygons of  $H_{\text{cris}}^2(X/W)$  in Exercise 3.10 in terms of some parameter  $h$ .

**Proposition 6.17.** *Let  $X$  be a K3 surface over a perfect field  $k$  of positive characteristic and let  $h$  be the height of its formal Brauer group.*

(1) *If  $h < \infty$ , then the slopes and multiplicities of  $H_{\text{cris}}^2(X/W)$  are as follows*

$$\begin{array}{cccc} \text{slope} & 1 - \frac{1}{h} & 1 & 1 + \frac{1}{h} \\ \text{multiplicity} & h & 22 - h & h \end{array}$$

(2) *If  $h = \infty$ , then  $H_{\text{cris}}^2(X/W)$  is of slope 1 with multiplicity 22.*

*In particular,  $h$  determines the Newton polygon of  $H_{\text{cris}}^2(X/W)$ .*

PROOF. By Proposition 6.15,  $\Phi_X^2$  is pro-representable by a 1-dimensional formal group law  $\widehat{\text{Br}}_X$ , and let  $h$  be its height.

If  $h = \infty$ , then  $D\widehat{\text{Br}}_X \otimes_W K = 0$  by Examples 6.12, and so, there are no slopes of  $H_{\text{cris}}^2(X/W)$  less than 1. Thus, by Exercise 3.10,  $H_{\text{cris}}^2(X/W)$  is of slope 1 with multiplicity 22.

If  $h < \infty$ , then the considerations of Example 6.16 show that the only slope less than 1 of  $H_{\text{cris}}^2(X/W)$  is equal to  $1 - \frac{1}{h}$ . By Exercise 3.10, the Newton polygon of  $H_{\text{cris}}^2(X/W)$  has the stated slopes and multiplicities.  $\square$

**Exercise 6.18.** Let  $X$  be a K3 surface over an algebraically closed field  $k$  of positive characteristic, let  $h$  be the height of its formal Brauer group, and let  $\rho$  be the rank of its Néron–Severi group.

(1) In case  $h < \infty$  use the previous proposition and the fact that the image of the crystalline Chern class has slope 1 to deduce the *Igusa–Artin–Mazur inequality*

$$\rho \leq 22 - 2h.$$

(In fact, this inequality can be generalized to arbitrary smooth projective varieties, see [Ill79a, Proposition (II.5.12)].) Since  $X$  is projective, we have  $\rho \geq 1$ , and therefore,  $h = 11$  is impossible.

(2) In case  $\rho = 22$  show that  $h = \infty$ . In fact, the Tate conjecture (see Conjecture 4.5 and Theorem 4.8) predicts the equivalence of  $\rho = 22$  and  $h = \infty$ .

- (3) Assuming the Tate conjecture, show that there exist no K3 surfaces with  $\rho = 21$  (this observation is due to Swinnerton-Dyer, see [Ar74a]).

We refer to [Ha78, Appendix B] for more results on formal group laws arising in algebraic geometry, as well as further references.

## 7. UNIRATIONAL K3 SURFACES

Over algebraically closed fields, a curve is rational if and only if it is unirational by Lüroth's theorem. By a theorem of Castelnuovo, this is also true for surfaces in characteristic zero. On the other hand, Zariski constructed examples of unirational surfaces that are not rational over algebraically closed fields of positive characteristic. Since the characterization of unirational surfaces in positive characteristic is still unclear, K3 surfaces provide an interesting testing ground.

The first examples of unirational K3 surfaces were constructed by Rudakov and Shafarevich in characteristic 2 and 3, as well as by Shioda in arbitrary characteristic. Also, Artin and Shioda showed that unirational K3 surfaces are supersingular (using different notions of supersingularity, now known to be equivalent by the established Tate-conjecture). Conversely, Artin, Rudakov, Shafarevich, and Shioda conjectured that supersingular K3 surfaces are unirational, which would give a cohomological characterization of unirationality.

In this section, we will use the formal Brauer group to associate to a supersingular K3 surface  $X$  over an algebraically closed field  $k$  of positive characteristic a 1-dimensional family  $\mathcal{X} \rightarrow \operatorname{Spec} k[[t]]$  of supersingular K3 surfaces with special fiber  $X$ , such that generic and special fiber are related by a *purely inseparable isogeny*. In particular, the generic fiber is unirational if and only if the special fiber is. Then, we fill up the moduli space of supersingular K3 surfaces with such families, which implies that every two supersingular K3 surfaces are purely inseparably isogenous. Since Shioda established the existence of some unirational K3 surfaces in every positive characteristic, the existence of these isogenies implies that all of them are, thereby establishing the Artin–Rudakov–Shafarevich–Shioda conjecture.

**7.1. The Lüroth problem.** We start by recalling some definitions and classical facts concerning the (uni-)rationality of curves and surfaces. In this section, the ground field  $k$  is always assumed to be algebraically closed, in order to avoid the distinction between (uni-)rationality over  $k$  and  $\bar{k}$ .

**Definition 7.1.** An  $n$ -dimensional variety  $X$  over an algebraically closed field  $k$  is called *unirational*, if there exists a dominant and rational map

$$\mathbb{P}_k^n \dashrightarrow X.$$

Moreover,  $X$  is called *rational* if this map can be chosen to be birational.

Equivalently,  $X$  is rational if and only if  $k(X) \cong k(t_1, \dots, t_n)$ , and  $X$  is unirational if and only if  $k(X) \subseteq k(t_1, \dots, t_n)$  of finite index. In particular, rational varieties are unirational, which motivates the following question.

**Question 7.2** (Lüroth). Are unirational varieties rational?



Lüroth showed that this is true for curves, which nowadays is an easy consequence of the Riemann–Hurwitz formula, see for example, [Har77, Example IV.2.5.5]. Next, Castelnuovo (characteristic zero) and Zariski (positive characteristic) showed that a surface is rational if and only if  $h^1(\mathcal{O}_X) = h^0(\omega_X^{\otimes 2}) = 0$ , that is, we have a cohomological criterion for rationality. Using this, one can show that Lüroth’s question also has a positive answer for surfaces in characteristic zero. We refer to [BHPV, Theorem VI.3.5] or [Be96, Chapter V] for details and proofs in characteristic zero, and to [Li13, Section 9] for a discussion in positive characteristic. In particular, a K3 surface in characteristic zero cannot be unirational.

On the other hand, Zariski [Za58] gave examples of unirational surfaces over algebraically closed fields of positive characteristic that are not rational, see also [Li13, Section 9] for more examples and discussion, as well as [L-S09] for results that show that unirationality is quite common even among simply connected surfaces of general type in positive characteristic.

Finally, there are 3-dimensional unirational varieties over algebraically closed fields of characteristic zero that are not rational by results of Iskovskih and Manin [I-M71], Clemens and Griffiths [C-G72], as well as Artin and Mumford [A-M72].

**7.2. Unirational and supersingular surfaces.** By Castelnuovo’s theorem, a surface in characteristic zero is rational if and only if it is unirational. Although this is not true in positive characteristic, we have at least the following necessary condition for unirationality.

**Theorem 7.3** (Shioda  $+\varepsilon$ ). *Let  $X$  be a smooth, proper, and unirational surface over an algebraically closed field  $k$  of positive characteristic.*

- (1) *The Picard rank  $\rho$  is equal to the second Betti number  $b_2$ , that is,  $X$  is Shioda-supersingular.*
- (2) *The crystal  $H_{\text{cris}}^2(X/W)$  is of slope 1, that is,  $X$  is Artin-supersingular.*

PROOF. Assertion (1) is shown in [Sh74a, Section 2]. Assertion (2) follows from (1), since  $c_1(\text{NS}(X)) \otimes W$  defines an  $F$ -crystal of slope 1 and rank  $\rho$  inside  $H_{\text{cris}}^2(X/W)$ , which is of rank  $b_2$ , see the discussion in Section 4.2 and the proof of Theorem 4.8.  $\square$

The proof also shows that we have the following implications for surfaces in positive characteristic

$$\text{unirational} \Rightarrow \text{Shioda-supersingular} \Rightarrow \text{Artin-supersingular}.$$

For K3 surfaces in odd characteristic, both notions of supersingularity are equivalent by the Tate conjecture, see Theorem 4.8. Thus, it is natural to ask whether all converse implications hold, at least, for K3 surfaces.

**Conjecture 7.4** (Artin, Rudakov, Shafarevich, Shioda). *A K3 surface is unirational if and only if it is supersingular.*

Let us note that Shioda [Sh77a, Proposition 5] gave an example of a Shioda-supersingular Godeaux surface that is not unirational. However, his examples have a fundamental group of order 5 and the non-unirationality of them is related to

congruences of the characteristic modulo 5. Therefore, in loc. cit. he asks whether simply connected and supersingular surfaces are unirational, which is wide open. Coming back to K3 surfaces, Conjecture 7.4 holds in the following cases:

**Theorem 7.5.** *Let  $X$  be a K3 surface characteristic  $p > 0$ . Then,  $X$  is unirational in the following cases:*

- (1)  $X$  is the Kummer surface  $\text{Km}(A)$  for a supersingular Abelian surface and  $p \geq 3$  by Shioda [Sh77b].
- (2)  $X$  is Shioda-supersingular and
  - (a)  $p = 2$  by Rudakov and Shafarevich [R-S78].
  - (b)  $p = 3$  and  $\sigma_0 \leq 6$  by Rudakov and Shafarevich [R-S78].
  - (c)  $p = 5$  and  $\sigma_0 \leq 3$  by Pho and Shimada [P-S06].
  - (d)  $p \geq 3$  and  $\sigma_0 \leq 2$  by (1), since these surfaces are precisely Kummer surfaces for supersingular Abelian surfaces (see Examples 4.10).

In particular, we have examples in every positive characteristic that support Conjecture 7.4. Let us comment on the methods of proof:

- (1) Shioda showed (1) by dominating Kummer surfaces by Fermat surfaces, that is, surfaces of the form  $\{x_0^n + \dots + x_3^n = 0\} \subset \mathbb{P}_k^3$ , and explicitly constructed unirational parametrizations of these latter surfaces in case there exists a  $\nu$  such that  $p^\nu \equiv -1 \pmod n$ , see [Sh74a].
- (2) Rudakov and Shafarevich [R-S78] showed their unirationality result using quasi-elliptic fibrations, which can exist only if  $2 \leq p \leq 3$ .

We refer to [Li13, Section 9] for more details and further references.

**7.3. Moving torsors.** An interesting feature of supersingular K3 surfaces is that all of them come with elliptic fibrations. In this section we will show that those that admit an elliptic fibration with section admit very particular 1-dimensional deformations: namely, the generic and special fiber are related by purely inseparable isogenies, which implies that one is unirational if and only if the other one is.

**Definition 7.6.** A genus-1 fibration from a surface is a proper morphism

$$f : X \rightarrow B$$

from a normal surface  $X$  onto a normal curve  $B$  with  $f_*\mathcal{O}_X \cong \mathcal{O}_B$  such that the generic fiber is integral of arithmetic genus 1. In case the geometric generic fiber is smooth, the fibration is called *elliptic*, otherwise it is called *quasi-elliptic*. In both cases, the fibration is called *Jacobian* if it admits a section.

If  $X \rightarrow B$  is a K3 surface together with a fibration, then  $B \cong \mathbb{P}^1$  for otherwise, the Albanese map of  $X$  would be non-trivial, contradicting  $b_1(X) = 0$ . Moreover, if  $F$  is a fiber, then  $F^2 = 0$ , and since  $\omega_X \cong \mathcal{O}_X$ , the adjunction formula yields  $2p_a(F) - 2 = F^2 + K_X F = 0$ , that is, the fibration is of genus 1. If the geometric generic fiber is singular, then, by Tate's theorem [Ta52], the characteristic  $p$  of the ground field  $k$  satisfies  $2 \leq p \leq 3$ . In particular, if  $p \geq 5$ , then every genus-1 fibration is generically smooth, that is, elliptic.

**Theorem 7.7.** *Let  $X$  be a supersingular K3 surface in odd characteristic  $p$ , or, Shioda-supersingular if  $p = 2$ .*

- (1)  *$X$  admits an elliptic fibration.*
- (2) *If  $p = 2$  or  $p = 3$  and  $\sigma_0 \leq 6$ , then  $X$  admits a quasi-elliptic fibration.*
- (3) *If  $p \geq 5$  and  $\sigma_0 \leq 9$ , then  $X$  admits a Jacobian elliptic fibration.*

PROOF. Since indefinite lattices of rank  $\geq 5$  contain non-trivial isotropic vectors, the Néron–Severi lattice  $\text{NS}(X)$  of a supersingular K3 surface contains a class  $0 \neq E$  with  $E^2 = 0$ . By Riemann–Roch on K3 surfaces,  $E$  or  $-E$  is effective, say  $E$ . Then, the Stein factorization of  $X \dashrightarrow \text{Proj } \bigoplus_n H^0(X, \mathcal{O}_X(nE))$  eventually gives rise to a (quasi-)elliptic fibration, see, for example, [R-S81, Section 3] or the proof of [Ar74a, Proposition 1.5] for details. Moreover, if  $X$  admits a quasi-elliptic fibration, it also admits an elliptic fibration, see [R-S81, Section 4].

Assertion (2) follows from the explicit classification of Néron–Severi lattices of supersingular K3 surfaces and numerical criteria for the existence of a quasi-elliptic fibration, see [R-S81, Section 5].

Assertion (3) follows again from the explicit classification of Néron–Severi lattices of supersingular K3 surfaces, see [Li15, Proposition 3.9] and [Li15, Remark 3.11].  $\square$

Let us now explain how to deform Jacobian elliptic fibrations on *supersingular* K3 surfaces to non-Jacobian elliptic fibrations: thus, let  $X \rightarrow \mathbb{P}^1$  be a Jacobian elliptic fibration from a K3 surface. Contracting the components of the fibers that do not meet the zero section, we obtain the *Weierstraß model*  $X' \rightarrow \mathbb{P}^1$ . This fibration has irreducible fibers,  $X'$  has at worst rational double point singularities, and we let  $A \rightarrow \mathbb{P}^1$  be the smooth locus of  $X' \rightarrow \mathbb{P}^1$ . Then,  $A \rightarrow \mathbb{P}^1$  is a relative group scheme - more precisely, it is the identity component of the Néron model of  $X \rightarrow \mathbb{P}^1$ . Now, one can ask for commutative diagrams of the form

$$\begin{array}{ccccccc} X & \supseteq & A & \rightarrow & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}_k^1 & = & \mathbb{P}_k^1 & \rightarrow & \mathbb{P}_S^1 \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & = & \text{Spec } k & \rightarrow & S \end{array}$$

where  $S$  is the (formal) spectrum of a local complete and Noetherian  $k$ -algebra with residue field  $k$ , the right squares are Cartesian, where  $\mathcal{A} \rightarrow \mathbb{P}_S^1$  is a family of elliptic fibrations with special fiber  $A \rightarrow \mathbb{P}_k^1$ , and where all elliptic fibrations are torsors (principal homogeneous spaces) under the Jacobian elliptic fibration  $A \rightarrow \mathbb{P}_k^1$ . To bound the situation and in order to algebraize, we assume that these *moving torsors* come with a relative invertible sheaf on  $\mathcal{A} \rightarrow \mathbb{P}_S^1$  that is of some fixed degree  $n$ . In [Li15, Section 3.1], we showed that such families correspond to  $n$ -torsion elements in the formal Brauer group, that is, elements in the Abelian group

$$\widehat{\text{Br}}_X(S)[n].$$

To obtain non-trivial moving torsors over  $S := \text{Spec } k[[t]]$ , this group must be non-zero, and for this to be the case, it simply follows from the fact that  $\widehat{\text{Br}}_X$  is

a 1-dimensional formal group law, that  $k$  must be of positive characteristic  $p$ , that  $\widehat{\mathrm{Br}}_X$  must be isomorphic to  $\widehat{\mathbb{G}}_a$ , and that  $p$  has to divide  $n$ , see [Li15, Lemma 3.3]. In particular,  $X$  must be a supersingular K3 surface. In case  $p = n$ , one can even find a degree- $p$  multisection  $D \rightarrow Y_S$ , that is purely inseparable over the base  $\mathbb{P}_S^1$ , see [Li15, Proposition 3.5]. Next, we use these considerations to construct a family of supersingular K3 surfaces.

**Theorem 7.8.** *Let  $X$  be a supersingular K3 surface in characteristic  $p \geq 5$ .*

- (1) *If  $\sigma_0 \leq 9$ , then  $X$  admits a Jacobian elliptic fibration.*
- (2) *Associated to a Jacobian elliptic fibration on  $X$ , there exists a smooth projective family of supersingular K3 surfaces*

$$\begin{array}{ccc} X & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec} k & \rightarrow & \mathrm{Spec} k[[t]]. \end{array}$$

*This family has the following properties:*

- (a) *The Artin invariants of special and geometric generic fiber satisfy*

$$\sigma_0(\mathcal{X}_{\overline{\eta}}) = \sigma_0(X) + 1.$$

*In particular, this family has non-trivial moduli.*

- (b) *There exist dominant and rational maps*

$$\mathcal{X}_{\overline{\eta}}^{(1/p)} \dashrightarrow X \times_k \overline{\eta} \dashrightarrow \mathcal{X}_{\overline{\eta}}^{(p)},$$

*both of which are purely inseparable of degree  $p^2$ .*

The idea is to compactify the moving torsor associated to the Jacobian elliptic fibration  $X \rightarrow \mathbb{P}^1$  that arises from a nontrivial  $p$ -torsion element of  $\widehat{\mathrm{Br}}_X(S)$  with  $S = \mathrm{Spec} k[[t]]$ . After resolving the rational double point singularities in families, we obtain a family  $\mathcal{X} \rightarrow \mathbb{P}_S^1$  of supersingular K3 surfaces with special fiber  $X$ . Specialization induces an injection of Néron–Severi groups  $\mathrm{NS}(\mathcal{X}_{\overline{\eta}}) \rightarrow \mathrm{NS}(X)$ , whose cokernel is generated by the class of the zero-section of  $X \rightarrow \mathbb{P}^1$ . From this, the assertion on Artin invariants follows, and we refer to [Li15, Theorem 3.6] for details. By [Li15, Proposition 3.5], the elliptic fibration on the generic fiber admits a degree- $p$  multisection  $D$  that is purely inseparable over the base. Then, the assertions on dominant and rational maps follow from base-changing the family  $\mathcal{X}_{\overline{\eta}} \rightarrow \mathbb{P}_{\overline{\eta}}^1$  to  $D \rightarrow \mathbb{P}_{\overline{\eta}}^1$ , which trivializes the moving torsor, see [Li15, Theorem 3.6].

Spreading out the family  $\mathcal{X} \rightarrow S$  of Theorem 7.8 to some curve of finite type over  $k$ , and using the theorem of Rudakov and Shafarevich on potential good reduction of supersingular K3 surfaces (see the remarks after Corollary 5.6), we obtain a smooth and projective family  $\mathcal{Y} \rightarrow B$ , where  $B$  is a smooth and projective curve over  $k$ , with  $X$  as some fiber, and  $\mathcal{X}$  as fiber over the completion  $\widehat{\mathcal{O}}_{B,\eta}$ . Associating to such a family their rigidified K3 crystals, we obtain the following statement about moduli spaces of rigidified K3 crystals.

**Theorem 7.9.** *Let  $N$  and  $N_+$  be the supersingular K3 lattices in characteristic  $p \geq 5$  of Artin invariants  $\sigma_0$  and  $\sigma_0 + 1$ , respectively. Then, there exists a surjective*

*morphism of moduli spaces of rigidified K3 crystals*

$$\mathcal{M}_{N_+} \longrightarrow \mathcal{M}_N,$$

*which comes with a section, and which turns  $\mathcal{M}_{N_+}$  into a  $\mathbb{P}^1$ -bundle over  $\mathcal{M}_N$ .*

We note that the fibers of this fibration correspond to the moving torsor families (at least, an open and dense subset does), and refer to [Li15, Theorem 4.3] and [Li15, Theorem 4.5] for details.

As explained in and after Examples 4.20, there is only one supersingular K3 surface with Artin invariant  $\sigma_0 = 1$ . In particular, by Theorem 7.8 and Theorem 7.9 together with an induction on Artin invariants, we can first relate every supersingular K3 surface to this  $\sigma_0 = 1$ -K3 surface via dominant rational maps, and ultimately obtain the following structure result for supersingular K3 surfaces.

**Theorem 7.10.** *Let  $X$  and  $Y$  be supersingular K3 surfaces in characteristic  $p \geq 5$ . Then, there exist dominant and rational maps*

$$X \dashrightarrow Y \dashrightarrow X$$

*which are purely inseparable, that is, the surfaces are purely inseparably isogenous.*

By a theorem of Shioda [Sh77b], supersingular Kummer surfaces in odd characteristic are unirational (see also Theorem 7.5 above), and combining this with the previous theorem, this implies Conjecture 7.4.

**Theorem 7.11.** *Supersingular K3 surfaces in characteristic  $p \geq 5$  are unirational.*

In particular, a K3 surface in characteristic  $p \geq 5$  is supersingular in the sense of Artin if and only if it is supersingular in the sense of Shioda if and only if it is unirational.

**7.4. Unirationality of moduli spaces.** It follows from Theorem 4.16 (but see also Theorem 7.9), that the moduli space of  $N$ -rigidified K3-crystals  $\mathcal{M}_N$  is rational. Thus, also the moduli space  $\mathcal{P}_N$  of  $N$ -rigidified K3-crystals together with ample cones is in some sense rational (this space is neither separated nor of finite type, but obtained by glueing open pieces of  $\mathcal{M}_N$ ). Moduli spaces of polarized K3 surfaces are much better behaved, see Theorem 2.10. In fact, constructing families of supersingular K3 surfaces using the formal Brauer group (similar to the moving torsor construction above, but more general), the supersingular loci inside moduli spaces of polarized K3 surfaces are rationally connected - this is a forthcoming result of Lieblich, see [Lieb13, Section 9] for announcements of some of these results.

## 8. BEYOND THE SUPERSINGULAR LOCUS

In Section 2.4, we introduced and discussed the moduli stack  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  of degree- $2d$  primitively polarized K3 surfaces over  $\mathbb{F}_p$ . In the previous sections, we focused on supersingular K3 surfaces, that is, K3 surfaces, whose formal Brauer groups are of infinite height. In this final section, we collect and survey a couple of results on

this moduli space beyond the supersingular locus. We stress that these results are just a small outlook, as well as deliberately a little bit sketchy.

**8.1. Stratification.** Associated to a K3 in positive characteristic  $p$ , we associated the following discrete invariants:

- (1) The height  $h$  of the formal Brauer group, which satisfies  $1 \leq h \leq 10$  or  $h = \infty$ , see Proposition 6.17 and Exercise 6.18.
- (2) If  $h = \infty$  then  $\text{disc}(\text{NS}(X)) = -p^{2\sigma_0}$  for some integer  $1 \leq \sigma_0 \leq 10$ , the Artin-invariant, see Definition 4.12.

These invariants allow us to define the following loci (just as a set of points, we do not care about scheme structures at the moment) inside the moduli space  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  of degree- $2d$  polarized K3 surfaces

$$\begin{aligned} \mathcal{M}_i &:= \{ \text{surfaces with } h \geq i \} \\ \mathcal{M}_{\infty, i} &:= \{ \text{surfaces with } h = \infty \text{ and } \sigma_0 \leq i \} . \end{aligned}$$

Thus, at least on the set-theoretical level, we obtain inclusions

$$\begin{aligned} \mathcal{M}_{2d, \mathbb{F}_p}^\circ &= \mathcal{M}_1 \supset \dots \supset \mathcal{M}_{10} \supset \mathcal{M}_\infty \\ &\quad \parallel \\ &\quad \mathcal{M}_{\infty, 10} \supset \mathcal{M}_{\infty, 9} \supset \dots \supset \mathcal{M}_{\infty, 1} . \end{aligned}$$

This stratification was introduced by Artin [Ar74a], and studied in detailed by Ogus [Og01]. It turns out that each  $\mathcal{M}_{i+1}$  is a closed subset of  $\mathcal{M}_i$  and that  $\mathcal{M}_{\infty, i}$  is closed in  $\mathcal{M}_{\infty, i+1}$ . For example, the first closedness assertion can be deduced easily from the following result [G-K00, Theorem 5.1].

**Proposition 8.1** (Katsura–van der Geer). *Let  $X$  be a K3 surface over an algebraically closed field of positive characteristic  $p$ . Then*

$$h = \min \{ n \geq 1 : (F : H^2(X, W_n(\mathcal{O}_X)) \rightarrow H^2(X, W_n(\mathcal{O}_X))) \neq 0 \} .$$

In fact, this generalizes to higher dimensions: if  $X$  is a Calabi–Yau variety of dimension  $n$ , then the height of the one-dimensional formal group law associated to  $\Phi_X^n$  (notation as in Section 6.2) can be characterized as in the previous proposition, and we refer to [G-K03, Theorem 2.1] for details.

**8.2. Stratification via Newton polygon.** By Proposition 6.17, the height  $h$  of the formal Brauer group of a K3 surface determines the smallest slope of the Newton polygon of the  $F$ -crystal  $H_{\text{cris}}^2$ . Moreover, by Exercise 3.10, the smallest slope determines this Newton polygon completely. In particular, the height stratification (the first part of the stratification introduced above)

$$\mathcal{M}_{2d, \mathbb{F}_p}^\circ = \mathcal{M}_1 \supset \dots \supset \mathcal{M}_{10} \supset \mathcal{M}_\infty$$

coincides with the stratification by the Newton polygon associated to the  $F$ -crystal  $H_{\text{cris}}^2$ . This stratification also illustrates Grothendieck’s theorem that the Newton polygon goes up under specialization.

**8.3. Stratification via F-zips.** Let  $(X, \mathcal{L})$  be a primitively polarized K3 surface over an algebraically closed field  $k$  of positive characteristic  $p$  such that  $p$  does not divide  $\mathcal{L}^2 =: 2d$ . The cup product induces a non-degenerate quadratic form on  $H_{\text{dR}}^2(X/k)$ . (Since we assumed  $p \nmid 2d$ , we have  $p \neq 2$  and thus, we do not have to deal with subtleties of quadratic forms in characteristic 2.) Then, we define the *primitive cohomology* to be

$$M := c_1(\mathcal{L})^\perp \subset H_{\text{dR}}^2(X/k).$$

We note that the condition  $p \nmid 2d$  ensures that  $c_1(\mathcal{L})$  is non-zero, and thus,  $M$  is a  $21$ -dimensional  $k$ -vector space. The Hodge and its conjugate filtration on  $H_{\text{dR}}^2(X/k)$  give rise to two filtrations  $C^\bullet$  and  $D_\bullet$  on  $M$ , and the Cartier isomorphism induces isomorphisms  $\varphi_n : (\text{gr}_C^n)^{(p)} \rightarrow \text{gr}_D^n$ , see Section 1.2. Next, the quadratic form on  $H_{\text{dR}}^2(X/k)$  induces a non-degenerate quadratic form  $\psi$  on  $M$ , and it turns out that the filtrations are orthogonal with respect to  $\psi$ . Putting this data together, we obtain an *orthogonal F-zip*

$$(M, C^\bullet, D_\bullet, \varphi_\bullet, \psi)$$

of filtration type  $\tau$  with  $\tau(0) = \tau(2) = 1$  and  $\tau(1) = 19$ , see Definition 1.3. We refer to [M-W04, Section 5] or [P-W-Z12] for  $F$ -zips with additional structure.

As already mentioned in Section 1.2 and made precise by [M-W04, Theorem 4.4],  $F$ -zips of a fixed filtration type form an Artin stack that has only a finite number of points. More precisely, orthogonal  $F$ -zips of type  $\tau$  as above are discussed in detail in [M-W04, Example (6.18)]. Let us sketch their results: if  $(V, \psi)$  is an orthogonal space of dimension  $21$  over  $\mathbb{F}_p$ , then  $SO(V, \psi)$  has a root system of type  $B_{10}$ . After a convenient choice of roots, and with appropriate identifications, the Weyl group  $W$  of  $SO(V, \psi)$  becomes a subgroup of the symmetric group  $\mathfrak{S}_{21}$  as follows

$$W \cong \{\rho \in \mathfrak{S}_{21} \mid \rho(j) + \rho(22 - j) = 22 \text{ for all } j\}.$$

We set  $W_J := \{\rho \in W \mid \rho(1) = 1\}$  and it is easy to see that the set of cosets  $W_J \backslash W$  consists of  $20$  elements. As shown in [M-W04, Example (6.18)], there exists a bijection between the set of isomorphism classes of orthogonal  $F$ -zips of type  $\tau$  over  $\overline{\mathbb{F}}_p$  and  $W_J \backslash W$ . Moreover, the Bruhat order on  $W$  induces a *total* order on this set of cosets, that is, we can find representatives  $\overline{w}_1 > \dots > \overline{w}_{20}$ . Using this bijection and the representatives, we define

$$\mathcal{M}^{(i)} := \{\text{surfaces whose associated orthogonal } F\text{-zip corresponds to } \overline{w}_i\}.$$

This gives a decomposition of  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  into  $20$  disjoint subsets. This decomposition is related to the stratification from Section 8.1 as follows

$$\begin{aligned} \mathcal{M}_i \setminus \mathcal{M}_{i+1} &= \mathcal{M}^{(i)} & \text{for } 1 \leq i \leq 10 \\ \mathcal{M}_{\infty, 21-i} \setminus \mathcal{M}_{\infty, 20-i} &= \mathcal{M}^{(i)} & \text{for } 11 \leq i \leq 20, \end{aligned}$$

where it is convenient to set  $\mathcal{M}_{11} := \mathcal{M}_\infty$ , and  $\mathcal{M}_{\infty, 0} := \emptyset$ . In particular, both decompositions eventually give rise to the same stratification of the moduli space. Again, we refer to [M-W04, Example (6.18)] for details.

**8.4. Singularities of the strata.** In [G-K00] and [Og01], Katsura, van der Geer, and Ogus found a beautiful description of the singularities of the height strata  $\mathcal{M}_i$ , which, by Section 8.2 coincide with the Newton-strata.

**Theorem 8.2** (Katsura–van der Geer, Ogus). *Let  $\{\mathcal{M}_i\}_{i \geq 1}$  be the height stratification of  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$ . Still assuming  $p \nmid 2d$ , we have an equality of sets*

$$(\mathcal{M}_i)^{\text{sing}} = \mathcal{M}_{\infty, i-1} \quad \text{for all } 1 \leq i \leq 10,$$

where  $-^{\text{sing}}$  denotes the singular locus of the corresponding height stratum.

**8.5. Cycle classes.** To state the next result, we let  $\pi : \mathcal{X} \rightarrow \mathcal{M}_{2d, \mathbb{F}_p}^\circ$  be the universal polarized K3 surface, and we still assume  $p \nmid 2d$ . Then, we define the *Hodge class* to be the first Chern class

$$\lambda_1 := c_1 \left( \pi_* \Omega_{\mathcal{X}/\mathcal{M}_{2d, \mathbb{F}_p}^\circ}^2 \right).$$

Let us recall that  $\mathcal{M}_{2d, \mathbb{F}_p}^\circ$  is 19-dimensional, and that each stratum of our stratification is of codimension 1 inside the next larger stratum. The following result [E-G11, Theorem A] describes the cycle classes of these strata in terms of the Hodge class. We note that the moduli spaces  $\mathcal{M}_{2d, \mathbb{F}_p}$  are non-complete, but that the formulas still make sense on an appropriate compactification.

**Theorem 8.3** (Ekedahl–van der Geer). *In terms of the Hodge class  $\lambda_1$ , the cycle classes of the strata inside  $\mathcal{M}_{2d, \mathbb{F}_p}$  are as follows*

$$\begin{aligned} [\mathcal{M}_i] &= (p-1)(p^2-1) \cdots (p^{i-1}-1) \lambda_1^{i-1} \\ [\mathcal{M}_\infty] &= \frac{1}{2} (p-1)(p^2-1) \cdots (p^{10}-1) \lambda_1^{10} \\ [\mathcal{M}_{\infty, i}] &= \frac{1}{2} \frac{(p^{2(11-i)}-1)(p^{2(12-i)}-1) \cdots (p^{20}-1)}{(p+1)(p^2+1) \cdots (p^i+1)} \lambda_1^{20-i} \end{aligned}$$

for  $1 \leq i \leq 10$ .

The appearance of the factor  $1/2$  is related to the fact that the formulas of [G-K00, Theorem 14.2 and Section 15] count the supersingular stratum doubly, see also [G-K01] and [E-G11].

Theorem 8.3 measures the “size” of these strata, and can be thought of as a generalization of a theorem of Deuring for elliptic curves: namely, in characteristic  $p$ , elliptic curves form a 1-dimensional moduli space over  $\mathbb{F}_p$ . The formal Picard group of an elliptic curve either has height 1 (ordinary elliptic curve) or height 2 (supersingular elliptic curve), see Example 6.10. Ordinary elliptic curves form an open and dense set, and thus, the number of supersingular elliptic curves in a fixed characteristic  $p$  is finite. Now, theorem of Deuring gives a precise answer: classically, it is phrased by saying that there are  $[p/12] + \varepsilon_p$  supersingular elliptic curves for some  $0 \leq \varepsilon_p \leq 2$  depending on the congruence class of  $p$  modulo 12, see, for example, [Si86, Theorem V.4.1]. However, if we count supersingular elliptic curves and weight each one of them with respect to their automorphism group (which can be thought of as counting them on the moduli stack rather than



the coarse moduli space) we obtain the following, much more beautiful formula

$$\sum_{E \text{ supersingular}/\cong} \frac{1}{\#\text{Aut}(E)} = \frac{p-1}{24}.$$

Theorem 8.3 is a generalization of this way of counting to K3 surfaces.

We refer to [E-G11] for more about the singularities of the strata, as well as to [G13] for irreducibility results of the strata.

**8.6. A Torelli theorem via Shimura varieties.** We end our survey with a very sketchy discussion of a Torelli theorem for K3 surfaces in positive characteristic beyond the supersingular ones.

For curves, Abelian varieties, K3 surfaces,... the classical period map associates to such a variety some sort of linear algebra data as explained at the end of Section 1.1. Over the complex numbers, this linear algebra data is parametrized as points inside some Hermitian symmetric domain modulo automorphisms. Thus, the period map can be interpreted as a morphism from the moduli space of these varieties to a Hermitian symmetric domain. In this setting, a Torelli theorem is the statement that this period map is an immersion, or, at least étale.

First, we set up some Shimura varieties, which will serve as the Hermitian symmetric domain modulo automorphisms: let  $U$  be the hyperbolic plane over  $\mathbb{Z}$ , and set  $N := U^{\oplus 3} \oplus E_8^{\oplus 2}$ . Thus, abstractly,  $N$  is isometric to  $H^2(X, \mathbb{Z})$  of a K3 surface with the cup-product pairing (Poincaré duality), also known as the *K3 lattice*. Let  $e, f$  be a basis for the first copy of  $U$  in  $N$ . Then, for  $d \geq 1$ , we define

$$L_d := \langle e - df \rangle^\perp \subseteq N,$$

which is modelled on the primitive cohomology  $P^2(X) := c_1(\mathcal{L})^\perp \subset H^2(X, \mathbb{Z})$  of a polarization  $\mathcal{L}$  of self-intersection  $2d$ . We note that  $G_d := SO(L_d)$  is a semi-simple algebraic group over  $\mathbb{Q}$ . Next, let  $K_{L_d} \subset G_d(\mathbb{A}_f)$  be the largest subgroup of  $SO(L_d)(\widehat{\mathbb{Z}})$  that acts trivially on the discriminant  $\text{disc}(L_d) := L_d^\vee/L_d$ . Finally, let  $Y_{L_d}$  be the space of oriented negative definite planes in  $L_d \otimes \mathbb{R}$ . Associated to this data, we have the Shimura variety  $\text{Sh}(L_d)$ . It is a smooth Deligne–Mumford stack over  $\mathbb{Q}$  such that, as complex orbifolds, its  $\mathbb{C}$ -valued points are given by the double quotient

$$\text{Sh}(L_d)(\mathbb{C}) = G_{L_d}(\mathbb{Q}) \backslash (Y_{L_d} \times G_d(\mathbb{A}_f)) / K_{L_d},$$

and we refer to [MP13, Section 4.1] for details.

Let us now return to K3 surfaces: in Section 2.4 we introduced the moduli space  $\mathcal{M}_{2d, \mathbb{Z}[\frac{1}{2d}]}^\circ$  of degree- $2d$  primitively polarized K3 surfaces over  $\mathbb{Z}[\frac{1}{2d}]$ . Rather than working with this moduli space, we will add *spin structures* first: let  $\mathcal{L}$  be a primitive polarization with  $\mathcal{L}^2 = 2d$ . Let  $P_\ell^2(X)$  denotes the primitive  $\ell$ -adic cohomology of  $X$ , that is is the orthogonal complement of  $c_1(\mathcal{L})$  inside  $H_{\text{ét}}^2(X, \mathbb{Z}_\ell)$ . For us, a spin structure is a choice of isomorphism

$$\det(P_\ell^2(X)) \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \det(L_d) \otimes \mathbb{Z}/2\mathbb{Z},$$

and we denote by  $\widetilde{\mathcal{M}}_{2d, \mathbb{Z}[\frac{1}{2d}]}^\circ$  the moduli space of primitively polarized K3 surfaces together with a choice of spin structure. Forgetting the spin structure induces a morphism

$$\widetilde{\mathcal{M}}_{2d, \mathbb{Z}[\frac{1}{2d}]}^\circ \rightarrow \mathcal{M}_{2d, \mathbb{Z}[\frac{1}{2d}]}^\circ,$$

which is étale of degree 2, and we refer to [MP13, Section 4.1] and [Ri06, Section 6] for details and precise definitions.

Now, over the complex numbers, there classical period map can be interpreted as a morphism

$$\iota_{\mathbb{C}} : \widetilde{\mathcal{M}}_{2d, \mathbb{C}} \rightarrow \mathrm{Sh}(L_d)_{\mathbb{C}}.$$

Obviously, the left side can be defined over  $\mathbb{Q}$  by considering families of K3 surface with polarization and spin structures over  $\mathbb{Q}$ . The right side possesses a canonical model over  $\mathbb{Q}$ . And then, as shown by Rizov [Ri05], the period map  $\iota_{\mathbb{C}}$  descends to a map  $\iota_{\mathbb{Q}}$  over  $\mathbb{Q}$ .

Thus, by what we have said above, the following result [MP13, Theorem 5] is a Torelli type theorem for K3 surfaces in positive and mixed characteristic

**Theorem 8.4** (Madapusi Pera). *There exists a regular integral model  $\mathcal{S}(L_d)$  for  $\mathrm{Sh}(L_d)$  over  $\mathbb{Z}[\frac{1}{2}]$  such that  $\iota_{\mathbb{Q}}$  extends to an étale map*

$$\iota_{\mathbb{Z}[\frac{1}{2}]} : \widetilde{\mathcal{M}}_{2d, \mathbb{Z}[\frac{1}{2}]}^\circ \rightarrow \mathcal{S}(L_d).$$

When adding level structures, one can even achieve a period map that is an open immersion [MP13, Corollary 4.15]. As explained in [MP13, Section 1], the construction of this map over  $\mathbb{Z}[\frac{1}{2d}]$  is essentially due to Rizov [Ri10], and another construction is due to Vasiu [Va]. Finally, let us also mention that Nygaard [Ny83b] proved a Torelli-type theorem for ordinary K3 surfaces using the theory of canonical lifts for such surfaces and then applying the Kuga–Satake construction.

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